AN ALGEBRAIC APPROACH TO HARMONIC POLYNOMIALS ON $S^3$

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Abstract 350 words maximum: (PLEASE TYPE)

In this thesis we are going to study harmonic polynomials on spheres, with the particular attention to the 3-sphere $S^3$. As a Lie group, the algebraic structure of $S^3 \cong SU(2)$ enables us to study harmonic polynomials from an algebraic point of view. A purely algebraic approach has several advantages, such as the possibilities to extend to more general matrix groups and work over general fields such as finite fields, and importantly to rational numbers. This algebraic approach allows us to construct an explicit basis for the space of harmonic polynomials on $S^3$ and describe them from a geometrical point of view. Finally, we give an explicit algebraic formula for zonal harmonic polynomials on $S^3$ which can be identified as characters of Lie group $S^3 \cong SU(2)$.

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Show me your ways, Lord,
teach me your paths.
Guide me in your truth and teach me,
for you are God my Savior,
and my hope is in you all day long.

– Psalm 25:4-5 –
This is dedicated for you, Papa.
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Abstract

In this thesis we are going to study harmonic polynomials on spheres, with the particular attention to the 3-sphere $S^3$. As a Lie group, the algebraic structure of $S^3 \cong \mathrm{SU}(2)$ enables us to study harmonic polynomials from an algebraic point of view. A purely algebraic approach has several advantages, such as the possibilities to extend to more general matrix groups and work over general fields such as finite fields, and importantly to rational numbers. This algebraic approach allows us to construct an explicit basis for the space of harmonic polynomials on $S^3$ and describe them from a geometrical point of view. Finally, we give an explicit algebraic formula for zonal harmonic polynomials on $S^3$ which can be identified as characters of Lie group $S^3 \cong \mathrm{SU}(2)$. 
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Chapter 0

Introduction

In this thesis we are going to study harmonic analysis on spheres, with particular
attention to the 3-sphere $S^3$. This sphere is particularly interesting because it
possesses a group structure, albeit a non-commutative one. In fact, $S^1$ and $S^3$
are the only two spheres that have a group structure. The algebraic structure
of $S^3$ enables us to talk about harmonic analysis on the sphere from a purely
algebraic point of view. This thesis will be mainly focused on the study of
harmonic polynomials, as we can use the fact that every function defined on a
sphere can be decomposed into a series of harmonic polynomials, which serve as
the building blocks of spherical functions.

The main aim is to introduce the reader to an algebraic way of describing
harmonic polynomials on $S^3$. We will be particularly interested in the zonal
harmonic polynomials – harmonic polynomials on $S^3$ that are invariant under
rotation. These polynomials are compelling because in Lie group theory, zonal
harmonic polynomials are characters of the group $S^3 \simeq SU(2)$.

This algebraic way of describing harmonic polynomials on $S^3$ connects nat-
urally with the theory of quaternions. There are already various studies about
spherical harmonic polynomials for a general sphere, using a standard analytical
point of view, such as Axler [6] and Müller [22], so one will have options to go
with the standard view or the algebraic one. However, since the standard view
often incorporates transcendental functions such as the square root function and
trigonometric functions, it is not always computationally effective, and does not
extend easily to other fields. The algebraic point of view that will be developed
here will extend to $SL_2$ and also often to orthogonal group of matrices. However,
that extension will not be discussed in this thesis and could be a suggestion for
future work. Moreover, one may be able to generalize this theory to work over general fields such as finite fields, and importantly to rational numbers.

This thesis will be developed as follows. Chapter 0 will be reserved as the introduction to the thesis to give the reader the general idea of where the direction of the thesis is heading. In Chapter 1, we will introduce the basic materials needed to explain the algebraic structure of $S^3$ by discussing the algebraic structure of $S^1$ first. In Chapter 2 we see that multiplication of complex numbers or quaternions can be interpreted geometrically as spherical rotations. We will see that incorporating quaternions will be useful to explain rotations on $S^2$, which has no group structure. In Chapter 3 we present the standard theory of harmonic functions. We will introduce first the concept of decomposition of general polynomials into homogeneous polynomials and finally into harmonic polynomials. The standard theory constructed later will be a decent tool to explain the decomposition of homogeneous polynomials into harmonic polynomials, as given below.

**Theorem 1** Any homogeneous polynomial $p(x) = p(x_1, x_2, \ldots, x_n) \in P_k(\mathbb{A}^n)$ can be uniquely written in the form

$$p = p_k + Q(x)p_{k-2} + \cdots + Q(x)^m p_{k-2m},$$

where $m = \left\lfloor \frac{k}{2} \right\rfloor$ and each $p_i \in H_i(\mathbb{A}^n)$; that is, $p_i$ is harmonic homogeneous polynomial of degree $i$.

At the end of Chapter 3, we will give an explicit basis for harmonic polynomials of degree $k$. It is important to note here that although we present the standard theory, we want to extract as much theory as possible to work in a more general algebraic setting. If a certain theory is possible for generalization over a more general field, we will make a distinction. For example, instead of working on Euclidean spaces $\mathbb{R}^n$, we turn our attention to $n$-dimensional affine space $\mathbb{A}^n$ over a general field, which might also be the field of rational numbers. There are, however, several steps in which we will resort to an analytical point of view. We will mention it whenever we do so.

In the last chapter, we will turn to our purely algebraic setting and extend the ideas developed by Wildberger in the preprint [31]. We will focus on harmonic polynomials on $S^3$ by first discussing harmonic polynomials on $S^2$. We will attempt to approach the study of harmonic polynomials on $S^2$ from an algebraic
point of view and see that the lack of group structure will still be a necessary reason for us to use transcendental functions in describing zonal harmonics. However, that is not the case in $S^3$. The highlight of this thesis will be the derivation of an algebraic version of zonal harmonic polynomials in $S^3$, given in the Theorem below.

**Theorem 2** Up to a scalar, there is a unique zonal harmonic polynomial of degree $k$, namely

$$p_k(x) = \sum_{l=0}^{k} \sum_{m=0}^{l} \sum_{n=0}^{m} (-1)^l l!(2l-2m)!(2m-2n)!(2n)! \frac{x^{2n} y^{2m-2n} z^{2l-2m} t^{k-2l}}{(2l+1)!(l-m)!(m-n)!n!}.$$

With the introduction of the complex factorial basis, we can reformulate zonal harmonic polynomials on $S^3$ into another form, which is given by this Theorem.

**Theorem 3** The zonal harmonic polynomials $p_k$ described in factorial basis is equivalent with

$$p_k = \sum_{l=0}^{k} \frac{(-1)^l}{(k)} \left( \sum_{m=0}^{l} (-1)^m \frac{r^m s^m u^{k-l-m} v^{l-m}}{(m-n)!} \right),$$

the zonal harmonic polynomials in complex factorial basis.

In Appendix A, we will present the standard theory of spherical harmonic decompositions on $L^2(S^{n-1})$, the space of all square-integrable functions on $S^{n-1}$. In Appendix B, we will also present our own version in the classical problem of integrating polynomials on a sphere, as given in this theorem.

**Theorem 4** If $p(x) = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ be any monomials of degree $m$ in $S^{n-1} \subset \mathbb{R}^n$, and $\mu$ is the normalised measure on $S^{n-1}$ defined in a way such that

$$\int_{S^{n-1}} d\mu = 1,$$

then

$$\int_{S^{n-1}} p(x) \ d\mu = \frac{(n-2)!!}{(m+n-2)!!} \prod_{i=1}^{n} (m_i - 1)!!$$

precisely when all of the $m_i$ are even integers, and zero otherwise.
0. Introduction

This classical problem has been studied extensively from an analytical point of view. See, for example, Baker [7, page 43] and Folland [17], which incorporates the gamma function $\Gamma$. While our result is a reformulation of Baker’s and Folland’s, the proof is our own work and the result is expressed in terms of rational function while still maintaining the symmetry.
Chapter 1

The Algebra of Complex Numbers and Quaternions

In this review chapter we will discuss the tools needed to equip both two spheres $S^1$ and $S^3$ with group structures. We review how the algebra of complex numbers can be used to model the unit circle and how we can associate a matrix to each complex number. Most of what we will do in the case of complex numbers will generalize to the case of quaternions, with the twist that the multiplication is not commutative.

1.1 The Complex Numbers $\mathbb{C}$

Let $F$ be any field for which $x^2 + 1 = 0$ has no solution. A complex number system, denoted by $\mathbb{C}$, consists of elements of the form $z = x + yi$ – where $x$ and $y$ belong to $F$ – which then will be called complex numbers. This is a quadratic field extension of $F$. The number $i$ which corresponds to the case where $x = 0$ and $y = 1$ satisfies $i^2 = -1$. It is important to note that the number $i \notin F$ from our assumption. This set of complex numbers $\mathbb{C}$ can be regarded as $F^2$, with the identification of $z = x + yi$ as $(x, y)$. Thus, $\mathbb{C}$ can be viewed as a vector space over $F$, with the addition and scalar multiplication defined as

\[
(x_1, y_1) + (x_2, y_2) \equiv (x_1 + x_2, y_1 + y_2),
\]

\[
\lambda(x_1, y_1) \equiv (\lambda x_1, \lambda y_1),
\]
for every \( x_1, x_2, y_1, y_2, \lambda \in \mathbb{F} \). Moreover, \( \mathbb{C} \) is an algebra with the multiplication defined as usual as

\[
(x_1, y_1) \cdot (x_2, y_2) \equiv (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).
\]

(1.1)

Note that the multiplication defined above is commutative, since

\[
(x_2, y_2) \cdot (x_1, y_1) = (x_2 x_1 - y_2 y_1, x_2 y_1 + x_1 y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) = (x_1, y_1) \cdot (x_2, y_2).
\]

If \( z = x + yi \), then \( x \) is called the real part of \( z \) and \( y \) is the imaginary part of \( z \), written \( \text{Re}(z) = x \) and \( \text{Im}(z) = y \), respectively. The conjugate of \( z = x + yi \) is \( \overline{z} = x - yi \), and we have that for any \( z_1, z_2 \in \mathbb{C} \),

\[
\overline{z_1 z_2} = \overline{z_1} \overline{z_2},
\]

for if \( z_1 = x_1 + y_1 i \) and \( z_2 = x_2 + y_2 i \), then

\[
\overline{z_1 z_2} = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i = (x_1 x_2 - y_1 y_2) - (x_1 y_2 + x_2 y_1) i = (x_1 - y_1 i) (x_2 - y_2 i) = \overline{z_1} \overline{z_2}.
\]

Also define the product of any complex number with its conjugate the quadrance of \( z \), denoted by \( Q(z) \). Thus,

\[
Q(z) = z \overline{z} = (x + yi) (x - yi) = x^2 + y^2.
\]

(1.2)

We can view this quadrance as a function from \( \mathbb{C} \) to \( \mathbb{F} \). From the commutativity of \( \mathbb{C} \) it then follows that the quadrance is a multiplicative function, i.e.

\[
Q(z_1 z_2) = Q(z_1) Q(z_2)
\]

for any \( z_1, z_2 \in \mathbb{C} \), since

\[
Q(z_1 z_2) = \overline{z_1 z_2} \overline{z_1 z_2} = \overline{z_1} \overline{z_2} \overline{z_1} \overline{z_2} = Q(z_1) Q(z_2).
\]
Remark 1 Note that this is a departure from the usual approach when the underlying field is $\mathbb{R}$, as we do not assume a square root that allows us to define $\|z\|$.

Corollary 1 For any $\lambda \in \mathbb{F}$ and $z \in \mathbb{C}$, we have that $Q(\lambda z) = \lambda^2 Q(z)$.

Define
\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and consider the set of matrices
\[
\tilde{\mathbb{C}} = \{ a1 + bi \mid a, b \in \mathbb{F} \}
\]
\[
= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{F} \right\}.
\]
This is a subalgebra of the algebra $M_2(\mathbb{F})$ over $\mathbb{F}$. Clearly
\[
1^2 = 1, \quad 1i = i1 = i, \quad \text{and} \quad i^2 = -1.
\]

Since the mapping
\[
a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}
\]
is an algebra isomorphism between $\mathbb{C}$ and $\tilde{\mathbb{C}}$, we can identify the complex number $a + bi$ with $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. This identification sometimes helps us in computing the product of several complex numbers. For example, if we want to compute the product of three complex numbers where the underlying field is the set of rational numbers $\mathbb{Q}$, given by $z_1 = -5 + 2i$, $z_2 = 1 - i$, and $z_3 = 3 + 2i$, we can compute the product of these three matrices in $M_2(\mathbb{Q})$:
\[
\begin{pmatrix} -5 & 2 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -23 & 15 \\ -15 & -23 \end{pmatrix},
\]
giving us $z_1 z_2 z_3 = -23 + 15i$.

A complex number $z = x + yi$ is called a unit complex number precisely when its quadrance is 1; that is, $Q(z) = x^2 + y^2 = 1$. We denote the set of pairs $(x, y) \in \mathbb{F}^2$ such that $x^2 + y^2 = 1$ by $S^1$, which will be called the unit circle.
in \( \mathbb{F}^2 \). It is worth to note that the set of all unit complex numbers \( S^1 \) forms a group under multiplication, for if \( z_1, z_2 \in S^1 \) then \( Q(z_1) = Q(z_2) = 1 \) so

\[
Q(z_1 z_2) = Q(z_1) Q(z_2) = 1,
\]

hence \( z_1 z_2 \in S^1 \). The associative property follows from the associativity of complex numbers, the identity element is \( 1 = 1 + 0i \); and for every \( z \in S^1 \), the inverse of \( z \) is \( \overline{z} \) since \( z \overline{z} = \overline{z} z = 1 \).

**Example 1** In the finite field \( \mathbb{F}_7 \), it can be checked that \( x^2 + 1 = 0 \) has no solution. The unit circle in this setting is then

\[
S^1 = \{(0,1),(1,0),(0,6),(6,0),(2,2),(2,5),(5,2),(5,5)\}.
\]

### 1.2 The Quaternions \( \mathbb{H} \)

The theory of quaternions was first developed by Sir William Rowan Hamilton, one of the most prominent Irish mathematicians in the 19th century. The motivation of the discovery of quaternions started from the desire to generalize complex numbers in a natural way. Hamilton first came up with the idea of a new algebraic structure whose elements consist of a real part and two distinct imaginary parts, which would then be called the Theory of Triplets. Just as complex numbers can be used to represent rotations in two-dimensional space, Hamilton hoped that this new structure could represent rotation in three-dimensional space. He worked for years and failed to produce such an algebraic structure. It then occurred to Hamilton during a walk in 1843 that instead of having two distinct imaginary parts, three distinct imaginary parts should be considered (see [3, Chapter 12]).

Again, let \( \mathbb{F} \) be any field which has no element \( x \) that satisfies \( x^2 + 1 = 0 \). Formally, we have the following definition of quaternions.

**Definition 1** The algebra of quaternions, denoted by \( \mathbb{H} \), is

\[
\langle 1, i, j, k \rangle = \{a1 + bi + cj + dk : a, b, c, d \in \mathbb{F}\},
\]
where the elements $i, j, k$ satisfy the following multiplication relations:

\[
\begin{array}{ccc}
\cdot & 1 & i & j & k \\
1 & 1 & i & j & k \\
i & i & -1 & k & -j \\
j & j & -k & -1 & i \\
k & k & j & -i & -1 \\
\end{array}
\] (1.4)

The multiplication table reads, for example, that $ij = k$ and $kj = -i$.

Compare this with the complex number system. We have, in addition to $i$, two more elements called $j$ and $k$ which also behave like $i$; that is, $j^2 = k^2 = -1$. Note that from the multiplication table above, we have that $ijk = k^2 = -1$. From the table it is also clear that the multiplication is non-commutative. See that, for example, $k = ij \neq ji = -k$. This is the difference between the complex numbers and quaternions. The non-commutativity of quaternions makes things considerably more difficult. When performing calculations, we have to be careful with the order. Throughout the text, the zero quaternion is denoted by simply 0.

The real part of $q = a + bi + cj + dk$, denoted by Re $(q)$, is $a$. Quaternions of the form $bi + cj + dk$ are called pure quaternions. We will also write $a + bi + cj + dk$ in short of $a1 + bi + cj + dk$. Similar in the complex number case, we can identify any quaternion $q = a + bi + cj + dk \in \mathbb{H}$ as a 4-tuple $(a, b, c, d) \in \mathbb{F}^4$. More interestingly, since $a + bi + cj + dk = (a + bi) + (c + di)j$, we can also identify $q$ as $(a + bi, c + di) \in \mathbb{C}^2$.

Let $q_1 = a_1 + b_1i + c_1j + d_1k$ and $q_2 = a_2 + b_2i + c_2j + d_2k$ be two quaternions, and take any $\lambda \in \mathbb{F}$. Define the addition and scalar multiplication in a similar fashion as the case of complex numbers by

\[
q_1 + q_2 \equiv (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k, \quad (1.5)
\]

\[
\lambda q_1 \equiv (\lambda a_1) + (\lambda b_1)i + (\lambda c_1)j + (\lambda d_1)k. \quad (1.6)
\]

As a consequence of the multiplication table in (1.4), we have that

\[
q_1 \cdot q_2 \equiv (a_3 + b_3i + c_3j + d_3k),
\]
1. The Algebra of Complex Numbers and Quaternions

where

\[ a_3 = a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2, \]
\[ b_3 = a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1, \]
\[ c_3 = a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1, \]
\[ d_3 = a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1. \]

From now on, we will always write \( q_1q_2 \) in short of \( q_1 \cdot q_2 \) for all \( q_1, q_2 \in \mathbb{H} \).

Given a quaternion \( q = a + bi + cj + dk \), we define the **conjugate** of \( q \) by

\[ \overline{q} = a - bi - cj - dk, \]

and the **quadrance** \( Q \) of \( q \) by

\[ Q(q) = q\overline{q} = (a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2. \] (1.8)

Also observe that \( q\overline{q} = a^2 + b^2 + c^2 + d^2 \), so any quaternion commutes with its conjugate. Note that we also have \( Q(\lambda q) = \lambda^2 Q(q) \) for every \( q \in \mathbb{H} \). From the equation (1.8) above, if we set

\[ q^{-1} \equiv \frac{1}{Q(q)}\overline{q}, \]

then it is apparent that

\[ qq^{-1} = q^{-1}q = 1. \]

### 1.3 Identifying Quaternions with Matrices

The associative and distributive properties of quaternions are not trivial, though. While these properties can be checked manually by using (1.5) and (1.7), let us introduce another way of viewing quaternions. Let \( q = a + bi + cj + dk \) be any arbitrary quaternion. Let \( 1, i, j, \) and \( k \) be respectively the following matrices in the algebra \( \mathbb{M}_2(\mathbb{C}) \) over \( \mathbb{F} \), the space of all \( 2 \times 2 \) matrices with complex entries:

\[
1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]
\[
j \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]
Note the difference between this $i$ and the previously defined $i$ in the Section 1.1. Define $\mathbb{H} = \langle 1, i, j, k \rangle$, the subalgebra of $M_2(\mathbb{C})$ over the field $\mathbb{F}$ spanned by the matrices $1$, $i$, $j$, and $k$. An arbitrary element of this subspace can be written as

$$a1 + bi + cj + dk = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

which is a matrix of the form

$$\begin{pmatrix} w_1 & w_2 \\ -\overline{w_2} & \overline{w_1} \end{pmatrix}$$

where $w_1 = a + bi$ and $w_2 = c + di$ are both complex numbers. This generalizes the identification of complex numbers as matrices, in the special case where the imaginary part of $w_1$ and $w_2$ are both zero.

Do note that in the above matrices, we have that

$$i^2 = j^2 = k^2 = ijk = -1 \quad (1.9)$$

and

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j. \quad (1.10)$$

Elements in $\mathbb{H}$ correspond to elements in $\mathbb{H}$ by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$ \hspace{1cm} (1.11)

One can check that this identification respects the relations given in (1.9) and (1.10), and the map (1.11) is an algebra isomorphism, so we can also think of elements in $\mathbb{H}$ as simply quaternions as well. This gives a demonstration that the multiplication of quaternions gives an associative and distributive algebra structure, as can be seen from $\mathbb{H}$. If $q = a + bi + cj + dk$, call its corresponding matrix in $\mathbb{H}$ by $M_q = a1 + bi + cj + dk$. This correspondence is especially useful when we are dealing with multiplication of quaternions, just like in the case of complex numbers. From (1.7), we see that if $q_1 = t_1 + x_1i + y_1j + z_1k$ and $q_2 = t_2 + x_2i + y_2j + z_2k$, then

$$M_{q_1q_2} = \begin{pmatrix} a_3 + b_3i & c_3 + d_3i \\ -c_3 + d_3i & a_3 - b_3i \end{pmatrix}$$

$$= \begin{pmatrix} t_1 + x_1i & y_1 + z_1i \\ -y_1 + z_1i & t_1 - x_1i \end{pmatrix} \begin{pmatrix} t_2 + x_2i & y_2 + z_2i \\ -y_2 + z_2i & t_2 - x_2i \end{pmatrix}$$

$$= M_{q_1} M_{q_2}.$$
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For example, if \(q_1 = 2 + 3i + 4j - 5k\), \(q_2 = -1 + 2i - 3j + 4k\), and \(q_3 = 4i - 2j\), then their product can be calculated simply by multiplying

\[
M_{q_1}M_{q_2}M_{q_3} = \begin{pmatrix} 2 + 3i & 4 - 5i \\ -4 - 5i & 2 - 3i \end{pmatrix} \begin{pmatrix} -1 + 2i & -3 + 4i \\ 3 + 4i & -1 - 2i \end{pmatrix} \begin{pmatrix} 4i & -2 \\ 2 & -4i \end{pmatrix}
\]

\[
= \begin{pmatrix} -72 + 88i & -64 + 124i \\ 64 + 124i & -72 - 88i \end{pmatrix}
\]

so that \(q_1q_2q_3 = -72 + 88i - 64j + 124k\).

By what we have discovered so far, we conclude that the quaternions satisfy the following properties, together make the set of quaternions an algebra.

1. **Addition**
   - Commutativity: \(q_1 + q_2 = q_2 + q_1\) for all \(q_1, q_2 \in \mathbb{H}\),
   - Associativity: \(q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3\) for all \(q_1, q_2, q_3 \in \mathbb{H}\),
   - Existence of identity: There exists \(0 \in \mathbb{H}\) such that \(q + 0 = 0 + q = q\) for all \(q \in \mathbb{H}\),
   - Existence of inverse element: For all \(q \in \mathbb{H}\) there exists \(-q \in \mathbb{H}\) such that \(q + (-q) = (-q) + q = 0\).

2. **Multiplication**
   - Associativity: \((q_1q_2)q_3 = q_1(q_2q_3)\) for all \(q_1, q_2, q_3 \in \mathbb{H}\),
   - Existence of identity: There exists \(1 \in \mathbb{H}\) such that \(q1 = 1q = q\) for all \(q \in \mathbb{H}\),
   - Existence of inverse element: For all \(q \in \mathbb{H}\) satisfying \(Q(q) \neq 0\), there exists \(q^{-1} \in \mathbb{H}\) such that \(qq^{-1} = q^{-1}q = 1\).

3. **Distributive law**
   - Left distributive law: \(q_1(q_2 + q_3) = q_1q_2 + q_1q_3\) for all \(q_1, q_2, q_3 \in \mathbb{H}\),
   - Right distributive law: \((q_1 + q_2)q_3 = q_1q_3 + q_2q_3\) for all \(q_1, q_2, q_3 \in \mathbb{H}\).
Another advantage of interpreting quaternions as matrices is that we can retrieve the real part of any quaternion as the trace of the corresponding matrix. Suppose \( \theta = \alpha + \beta i + \gamma j + \delta k \) is any quaternion, then we have

\[
a = \frac{1}{2} \text{tr} (M_\theta).
\]

(1.12)

For all matrices \( X \) with complex entries, we define \( X^* \) to be its **conjugate transpose** or Hermitian transpose; that is, the matrix obtained from the transpose \( X^T \) by taking the complex conjugate of all entries. If \( q = a + bi + cj + dk \), then

\[
M_q^* = \begin{pmatrix}
a + bi & c + di \\
-c + di & a - bi
\end{pmatrix}^* \\
= \begin{pmatrix}
a - bi & -c - di \\
c - di & a + bi
\end{pmatrix}
= M_{q^*}.
\]

Also observe that if \( q = a + bi + cj + dk \), then we have that

\[
Q(q) = a^2 + b^2 + c^2 + d^2
= \det \begin{pmatrix}
a + bi & c + di \\
-c + di & a - bi
\end{pmatrix}
= \det (M_q).
\]

(1.13)

This is useful when we want to prove that \( Q \) is multiplicative; i.e.

\[
Q(q_1q_2) = Q(q_1)Q(q_2)
\]

(1.14)

for all \( q_1, q_2 \in \mathbb{H} \), since

\[
Q(q_1q_2) = \det (M_{q_1q_2})
= \det (M_{q_1}M_{q_2})
= \det (M_{q_1}) \det (M_{q_2})
= Q(q_1)Q(q_2).
\]

If \( Q(q) = 1 \), then \( q \) is a **unit quaternion**. The case where \( Q(q) = 1 \) is particularly important because the set \( \{ q \in \mathbb{H} \mid Q(q) = 1 \} \) is closed under multiplication, as can be seen from (1.14). In fact, it can be easily verified that
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this set is a multiplicative group. The associativity follows from the associativity of general quaternions. The identity element is 1, and the inverse element for \( q \) is just \( \overline{q} \). We denote this set by \( S^3 \), the 3-sphere of unit quaternions. If the underlying field \( \mathbb{F} = \mathbb{R} \), we can think of \( S^3 \) as a manifold in \( \mathbb{R}^4 \). Since \( S^3 \) is a manifold and also a group, \( S^3 \) is a Lie group. Classical theory of Lie groups can be found in Duistermaat [13], Hall [19], or Chevalley [11].

Definition 2 A matrix \( X \in M_n(\mathbb{C}) \) is called unitary if \( X^* X = I_n \), where \( I_n \) is the identity matrix. The unitary group \( U(n) \) is the subgroup of \( GL(n, \mathbb{C}) \) which consists of unitary matrices. The special unitary group \( SU(n) \) is the subgroup of \( U(n) \) that consists of matrices of determinant 1.

Remark 2 Note that if \( X^* X = I_n \) then \( \det(X) = \pm 1 \) since determinant is multiplicative and \( \det(X^*) = \det(X) \).

We could then associate two spheres that we have just built by the following proposition.

Proposition 1 The group \( S^1 \) is isomorphic to \( U(1) \). The group \( S^3 \) is isomorphic to \( SU(2) \).

Proof. Note that \( U(1) \) consists of all matrices of the form \( (z) \) such that \((z)^*(z) = (1)\), which means that \( \overline{zz} = 1 \). This is precisely the set of all unit complex numbers \( S^1 \), so the said isomorphism is \( z \mapsto (z) \). For the second statement, note that there is a bijection between \( q = a + bi + cj + dk \) and \( M_q = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \). The map \( q \mapsto M_q \) is also a homomorphism since \( M_{\eta_1} M_{\eta_2} = M_{\eta_1 \eta_2} \) for all \( \eta_1, \eta_2 \in S^3 \). Finally, if \( q \in S^3 \) then \( Q(q) = 1 \). But we know that \( 1 = Q(q) = \det(M_q) \), and that

\[ M_q^* M_q = M_{\eta q} = M_1 = I_2, \]

proving that \( M_q \) is unitary. Hence, \( M_q \in SU(2) \). ■

Suppose \( q_1 = a_1 + b_1i + c_1j + d_1k \) and \( q_2 = a_2 + b_2i + c_2j + d_2k \), define the quaternion inner product \( \langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \) by

\[ \langle q_1, q_2 \rangle \equiv a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2. \]

From this, it is evident that \( Q(q) = \langle q, q \rangle \). We also have that

\[ \langle q_1, q_2 \rangle = \frac{1}{2} \text{tr} \left( M_{q_1}^* M_{q_2} \right) \]
and the proof is computational. This shows that the theory of quaternions is intimately connected with the algebra of complex matrices. More material about this can be found in Wildberger [32, Part c].
Chapter 2

Spherical Rotation

As mentioned in the Chapter 1, the group of unit complex numbers $S^1$ can be used to explain rotations in the plane. Quaternions were created by Hamilton to help explain rotations in three-dimensional space. In this chapter, we will see how these two algebraic structures are useful for explaining the concept of rotation. Throughout this thesis, let $A^n$ be the affine space over a field $F$, which is an algebraic generalization of Euclidean space. When the underlying field is $\mathbb{R}$, then $A^n$ is just the usual $n$-dimensional Euclidean space. The material in this chapter is inspired by Wildberger’s quaternion lectures [32, Part b, Part c].

2.1 Rotation in $S^1$

In this section we are going to talk about how complex numbers previously defined can be utilized to describe rotation in two-dimensional affine space $A^2$. Let $z = a + bi$ be any complex number. Write $[z]$ to be the line that passes through $z$ and the origin. The standard theory to measure how far away a particular point from another point, aside from distance, is the notion of angle. As the direction of this thesis suggests, we aim to generalize the work that we develop here to other algebraic settings, so rational functions are preferable to transcendental ones. For this reason, we will find a rational analog of angle. There are several choices and all are developed from rational trigonometry (see Wildberger [30]), but in this section we will stick with only one.

Define the turn of $z = a + bi$ to be the element $\tau(z) = \frac{b}{a} := ba^{-1} \in F$, for $a \neq 0$. It follows that the turn of $z$ depends only on $[z]$. Define, for $z \neq 0$, the
2. Spherical Rotation

linear operator $\phi_z$ on $\mathbb{C}$

$$\phi_z (r) = \frac{z^2 r}{Q(z)}, \ r \in \mathbb{C}. \quad (2.1)$$

From Corollary 1 we observe that

$$Q \left( \frac{z^2}{Q(z)} \right) = 1,$$

so this is a rotation on $S^1$ determined by the line $[z]$. Moreover, we also have that

$$\phi_{z_1} \circ \phi_{z_2} = \phi_{z_1 z_2} \quad (2.2)$$

and

$$\phi_{\lambda z} = \phi_z \quad (2.3)$$

for any $z_1, z_2, \ z \in \mathbb{C}$ and $\lambda \in \mathbb{F}$. Equation (2.3) suggests that rotation on $S^1$ can be indexed by lines $[z]$ in the two-dimensional affine space $\mathbb{A}^2$. The set of all one-dimensional subspaces in $\mathbb{A}^2$ is called the **projective line** $\mathbb{P}^1$. It includes the line $[z]$ where $z = a + bi$ and $a \neq 0$ where the turn is defined; and also $[z]$ where $z = bi$, which is called the **point at infinity**. Hence, the assignment

$$[z] \mapsto \phi_{[z]}$$

shows us that rotations on $S^1$ can be parametrized by $\mathbb{P}^1$. If we write $r = x + yi$ then the above mapping corresponds to the matrix

$$\frac{1}{Q(z)} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{Q(z)} \begin{pmatrix} (a^2 - b^2) x - 2aby & 2abx + (a^2 - b^2) y \\ -2abx - (a^2 - b^2) y & (a^2 - b^2) x - 2aby \end{pmatrix},$$

so that

$$\phi_z (r) = \frac{1}{a^2 + b^2} \left( (a^2 - b^2) x - 2aby + (2abx + (a^2 - b^2) y) \right).$$

If we identify $r$ by $\begin{pmatrix} x \\ y \end{pmatrix}$ then we can view the linear map $\phi_z$ as

$$r \mapsto P_z r$$

where

$$P_z = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix}. \quad (2.4)$$
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One can check that the matrix \( P_z \) above is orthogonal since the first and the second column are perpendicular with respect to the standard dot product, and that \((a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2\). The latter also implies that \( \det P_z = 1 \), so \( P_z \in \text{SO} (2) \), the group of \( 2 \times 2 \) orthogonal matrices with determinant 1.

Note that we can write (2.4) as

\[
P_z = \frac{1}{1 + u^2} \begin{pmatrix} 1 - u^2 & -2u \\ 1 + u^2 & 1 - u^2 \end{pmatrix}
\]

where \( u = \tau(z) \), the turn of \( z \), for such \( z \) so that \( \text{Re} (z) \neq 0 \). Point at infinity is associated with \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \). Hence, there is an association

\[
[z] \mapsto P_z [z]
\]

that sends a line \([z] \) in \( \mathbb{P}^1 \) to \( \text{SO} (2) \).

2.2 Describing Rotations by Quaternions

We want to show how quaternions connect to rotations of three-dimensional affine space \( \mathbb{A}^3 \). Let \( q = t + xi + yj + zk \) be a quaternion. Denote by \( v \) the imaginary part of \( q \), so that \( v = xi + yj + zk \) and we can write \( q = t + v \). We can picture \( q \) as a vector \((t, x, y, z) \in \mathbb{A}^4 \), so we can think of \( v \) as a vector \((x, y, z) \) in the three-dimensional space \( V \simeq \mathbb{A}^3 \) spanned by \( i, j, \) and \( k \); which is perpendicular to the \( t \)-axis under the usual Euclidean inner product. Now consider the unit sphere \( S^3 \) in the four-dimensional space \( \mathbb{A}^4 \). It meets the \( t \)-axis at \((1, 0, 0, 0) \) which corresponds to \( q_1 = 1 \), the \( x \)-axis at \((0, 1, 0, 0) \) which corresponds to \( q_2 = i \), the \( y \)-axis at \((0, 0, 1, 0) \) which corresponds to \( q_3 = j \), and the \( z \)-axis at \((0, 0, 0, 1) \) which corresponds to \( q_4 = k \).

The equator of this sphere is exactly the intersection between \( S^3 \) and \( V \), which corresponds to the case where \( t = 0 \). In this case, the equation becomes \( x^2 + y^2 + z^2 = 1 \) which, in the three-dimensional space \( V \), is nothing but 2-sphere \( S^2 \). Now, if \( q = t + v \) is any quaternion then we can think of \( t \) as the projection of \( q \) to the \( t \)-axis and \( v \) as the projection to \( V \). Also note that if we reflect the point \( q = t + v \) with respect to the \( t \)-axis then we will get the point \( t - v \) which is exactly \( \overline{q} \).
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For $q = t + v$ a nonzero quaternion, define, for any $r \in \mathbb{H}$,

$$
\phi_q(r) \equiv qrq^{-1} = \frac{qr\tau}{Q(q)}.
$$

This is a linear operator on $\mathbb{H}$ and has the following properties.

**Proposition 2** We have that

1. $\phi_q(1) = 1$,
2. If $w \in V$ then $\phi_q(w) \in V$,
3. $\phi_q(v) = v$.

**Proof.** Clearly,

$$
\phi_q(1) = \frac{q1\tau}{Q(q)} = \frac{q\tau}{Q(q)} = 1.
$$

That proves the first assertion. Next, if $w \in V$, then we want to show that $\text{Re} \left( \phi_q(w) \right) = 0$. Note from (1.12), we have that

$$
\text{Re} \left( \phi_q(w) \right) = \frac{1}{2} \text{tr} \left( M_{\phi_q(w)} \right).
$$

Since

$$
M_{\phi_q(w)} = \frac{1}{Q(q)} M_q M_w M_\tau
$$

and

$$
\text{tr} \left( M_{\phi_q(w)} \right) = \frac{1}{Q(q)} \text{tr} (M_q M_w M_\tau)
$$

$$
= \frac{1}{Q(q)} \text{tr} (M_\tau M_q M_w)
$$

$$
= \text{tr} (M_w)
$$

$$
= 0,
$$

it follows that $\text{Re} \left( \phi_q(w) \right) = 0$, so that $\phi_q(w) \in V$. Here we have used the fact that $\text{tr} (AB) = \text{tr} (BA)$ for any $2 \times 2$ complex matrices $A$ and $B$, with $A = M_q M_w$ and $B = M_\tau$.

Now, for the third proposition, note that $q = t + v$ commutes with $v$ since $v$ commutes with $t$ and $v$ itself. Hence,

$$
\phi_q(v) = \frac{qv\tau}{Q(q)} = \frac{q\tau v}{Q(q)} = v.
$$
as desired.

From the Proposition 2 above we conclude that \( \phi_q \) must be a rotation in \( V \cong \mathbb{A}^3 \) with \( \mathbf{v} \) being the axis of rotation. Now, define \([q]\) to be the line in \( \mathbb{A}^4 \) that passes through \( q \) and the origin. The set of all lines \([q]\) is called the projective space \( \mathbb{P}^3 \). Observe, as in the \( S^1 \) case, that

\[
\phi_{\lambda q} = \phi_q
\]

for any \( q \in \mathbb{H} \) and \( \lambda \in \mathbb{F} \). Thus it is sensible to make the association \([q] \mapsto \phi_{[q]}\) that tells us that rotations in three-dimensional space \( V \) can be parametrized by \( \mathbb{P}^3 \).

As a linear operator on the vector space \( \mathbb{H} \), the map \( \phi_q \) can be associated to a matrix \( A_q \). Let \( q = s + w_1i + w_2j + w_3k \) be a fixed, arbitrary quaternion and take any \( r = t + v_1i + v_2j + v_3k \in \mathbb{H} \). Note that

\[
\phi_q (r) \equiv q r q^{-1} = \frac{q \tau q}{Q(q)}
\]

corresponds to the matrix

\[
\frac{1}{Q(q)} M_q M_r M_q^* = \frac{1}{Q(q)} M_q M_r M_q^*,
\]

where

\[
M_q = \begin{pmatrix}
  s + w_1i & w_2 + w_3i \\
  -w_2 + w_3i & s - w_1i
\end{pmatrix}
\quad \text{and} \quad
M_r = \begin{pmatrix}
  t + v_1i & v_2 + v_3i \\
  -v_2 + v_3i & t - v_1i
\end{pmatrix}.
\]

If we perform the matrix multiplication above and switch back into the form \( a + bi + cj + dk \), then we will have that

\[
\begin{align*}
a &= t, \\
b &= \frac{1}{Q(q)} \left( (s^2 + w_1^2 - w_2^2 - w_3^2) v_1 + 2 (w_1 w_2 - s w_3) v_2 + 2 (w_1 w_3 + s w_2) v_3 \right), \\
c &= \frac{1}{Q(q)} \left( 2 (w_1 w_2 + s w_3) v_1 + (s^2 - w_1^2 + w_2^2 - w_3^2) v_2 + 2 (w_2 w_3 - s w_1) v_3 \right), \\
d &= \frac{1}{Q(q)} \left( 2 (w_1 w_3 - s w_2) v_1 + 2 (w_2 w_3 + s w_1) v_2 + (s^2 - w_1^2 - w_2^2 + w_3^2) v_3 \right).
\end{align*}
\]
We can encode that information more neatly. If we identify $r$ with \[
\begin{pmatrix}
t
v_1
v_2
v_3
\end{pmatrix},
\] then we can identify $\phi_q$ with the linear map \[
r \mapsto A_q r
\] where \[
A_q = \frac{1}{Q(q)} \begin{pmatrix}
Q(q) & 0 & 0 & 0 \\
0 & s^2 + w_1^2 - w_2^2 - w_3^2 & 2(w_1w_2 - sw_3) & 2(w_1w_3 + sw_2) \\
0 & 2(w_1w_2 + sw_3) & s^2 - w_1^2 + w_2^2 - w_3^2 & 2(w_2w_3 - sw_1) \\
0 & 2(w_1w_3 - sw_2) & 2(w_2w_3 + sw_1) & s^2 - w_1^2 - w_2^2 + w_3^2
\end{pmatrix}.
\] Clearly this leaves the $t$-axis fixed. Denote by $B_q$ the matrix \[
\frac{1}{Q(q)} \begin{pmatrix}
s^2 + w_1^2 - w_2^2 - w_3^2 & 2(w_1w_2 - sw_3) & 2(w_1w_3 + sw_2) \\
2(w_1w_2 + sw_3) & s^2 - w_1^2 + w_2^2 - w_3^2 & 2(w_2w_3 - sw_1) \\
2(w_1w_3 - sw_2) & 2(w_2w_3 + sw_1) & s^2 - w_1^2 - w_2^2 + w_3^2
\end{pmatrix}, \tag{2.5}
\] and observe that $B_{\lambda q} = B_q$ for all $q \in \mathbb{H}$ and $\lambda \in \mathbb{F}$. One can compute directly that the matrix $B_q$ is orthogonal and see that every $3 \times 3$ orthogonal matrix can be written as $B_q$ for some unit quaternion $q = s + w_1i + w_2j + w_3k$ (see [15, Chapter 9]). Moreover, $\det(B_q) = 1$ for all $q \in \mathbb{H}$. We call the group of $3 \times 3$ orthogonal matrices of determinant 1 $\text{SO}(3)$, and we close this section by the following result.

**Proposition 3** There is an isomorphism between $\mathbb{F}^3$ and $\text{SO}(3)$.

**Proof.** The mapping \[
[q] \mapsto B_{[q]} = \frac{1}{Q(q)} \begin{pmatrix}
s^2 + w_1^2 - w_2^2 - w_3^2 & 2(w_1w_2 - sw_3) & 2(w_1w_3 + sw_2) \\
2(w_1w_2 + sw_3) & s^2 - w_1^2 + w_2^2 - w_3^2 & 2(w_2w_3 - sw_1) \\
2(w_1w_3 - sw_2) & 2(w_2w_3 + sw_1) & s^2 - w_1^2 - w_2^2 + w_3^2
\end{pmatrix}
\] gives the desired isomorphism. \qed

## 2.3 Spherical Parametrization

In this section we are going to present the sphere parametrization from an algebraic point of view that works well in any field $\mathbb{F}$, provided the equation $x^2 + 1 = 0$
does not have a solution in \( \mathbb{F} \). At the end of this section, we will present the standard theory of parametrization when the underlying field is just \( \mathbb{R} \). We start from the simplest sphere, the unit circle \( S^1 \), before moving to \( S^2 \) and finally \( S^3 \). This is closely related to the rotations we saw earlier.

### 2.3.1 Stereographic Projections

Stereographic projection is a way to parametrize a sphere by projecting each point in the \((n-1)\)-dimensional sphere \( S^{n-1} \subset \mathbb{A}^n \) onto a hyperplane from a fixed point on the sphere, which would be called the pole. We shall illustrate this in the case where \( n = 2 \). Take the unit circle \( x^2 + y^2 = 1 \) centered at the origin and fix the point \((-1,0)\) as our pole. For each \( u \in \mathbb{F} \), let \([u]\) be the line that connects the pole \((-1,0)\) and the point \((0,u)\) in \( \mathbb{P}^1 \subset \mathbb{A}^2 \). This line will intersect the circle exactly once, so that each point on the circle except the pole will correspond to \((0,u)\) which can be identified as \( u \). To put it in another words, each \([u]\) will correspond to exactly one point on the unit circle with \((-1,0)\) removed, where \( u \in \mathbb{F} \). The point at infinity in \( \mathbb{P}^1 \) will correspond to the pole \((-1,0)\). We identify the point at infinity with \( u = \infty \).

For each \( u \in \mathbb{F} \), the line that passes through \((-1,0)\) and \((0,u)\) in \( \mathbb{A}^2 \) is given by the equation \( y = ux + u \). Substituting it into the equation \( x^2 + y^2 = 1 \) we get

\[
x^2 + (ux + u)^2 = 1
\]

which will give us the quadratic equation

\[
(1 + u^2) x^2 + 2ux + (u^2 - 1) = 0
\]

which factorizes as

\[
(x + 1) \left( (1 + u^2) x - (1 - u^2) \right) = 0.
\]

We get two values of \( x \), namely

\[
x = -1 \text{ or } x = \frac{1 - u^2}{1 + u^2}.
\]

The case where \( x = -1 \) corresponds to the point \((-1,0)\) which we know already. The other value of \( x \) will give the value of \( y \), so that the other intersection point
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is given by

\[
(x, y) = \left( \frac{1 - u^2}{1 + u^2}, u \left( \frac{1 - u^2}{1 + u^2} \right) + u \right)
\]

\[
= \left( \frac{1 - u^2}{1 + u^2}, \frac{2u}{1 + u^2} \right).
\]

To put it in another words,

\[
x(u) = \frac{1 - u^2}{1 + u^2} \quad \text{and} \quad y(u) = \frac{2u}{1 + u^2},
\]

where \( u \in \mathbb{F} \) give us a parametrization of the unit circle \( S^1 \) with the point \((-1, 0)\) removed. Thus if the field \( \mathbb{F} \) does not have the square root of \(-1\), then the map

\[
\tau : \mathbb{P}^1 \mapsto S^1
\]

where

\[
\tau([u]) = \begin{cases} 
\left( \frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2} \right) & \text{if } u \in \mathbb{F} \\
(-1,0) & \text{if } u = \infty
\end{cases}
\]

gives the desired parametrization of the whole sphere \( S^1 \) in terms of the extended, or projective line. We have the following result.

**Theorem 5** There is a 1 : 1 correspondence between the unit circle \( S^1 \) and \( \text{SO}(2) \).

**Proof.** The point \( \left( \frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2} \right) \) corresponds to the matrix

\[
\frac{1}{1 + u^2} \begin{pmatrix} 1 - u^2 & -2u \\ 1 + u^2 & 1 - u^2 \end{pmatrix}
\]

in \( \text{SO}(2) \) which is associated to the complex number \( z \) such that \( \text{Re}(z) \neq 0 \). The point \((-1, 0)\) corresponds to

\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},
\]

which is associated with the complex number \( z \) such that \( \text{Re}(z) = 0 \). This gives us the desired 1 : 1 correspondence. ■
2. Spherical Rotation

Example 2 If \( \mathbb{F} = \mathbb{F}_7 \), then \( u = 5 \) corresponds to the point

\[
(-24 \cdot 26^{-1}, 10 \cdot 26^{-1}) = (4 \cdot 5^{-1}, 3 \cdot 5^{-1}) = (4 \cdot 3, 3 \cdot 3) = (5, 2).
\]

More precisely, from Example 1, \( u = 0, 1, 2, 3, 4, 5, 6 \) correspond to the points \((1, 0), (0, 1), (5, 5), (2, 2), (2, 5), (5, 2), (0, 6), \) respectively. The point \((6, 0) = (-1, 0)\) corresponds to the point of infinity \( u = \infty \).

To deal with the 2-sphere, it is perhaps better to move our pole to the point \((0, 0, -1)\) so we can project every point on the sphere with the pole removed to the \(xy\)-plane, which is the subspace of \( \mathbb{A}^3 \) that consists of points of the form \((x, y, 0)\) where \( x, y \in \mathbb{F} \). The line connecting the pole and any arbitrary point \((u, v, 0)\) in the \(xy\)-plane is parametrized by

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
-1
\end{pmatrix} + \lambda \begin{pmatrix}
u \\
v \\
1
\end{pmatrix},
\]

where \( \lambda \in \mathbb{F} \). Substituting \( x = \lambda u, y = \lambda v, \) and \( z = \lambda - 1 \) into the equation \(x^2 + y^2 + z^2 = 1\), we get

\[(\lambda u)^2 + (\lambda v)^2 + (\lambda - 1)^2 = 1,
\]

which gives us the quadratic equation

\[(1 + u^2 + v^2) \lambda^2 - 2\lambda = 0.
\]

We get two values of \( \lambda \), namely

\[
\lambda = 0 \quad \text{or} \quad \lambda = \frac{2}{1 + u^2 + v^2},
\]

as long as \(u^2 + v^2 \neq -1\). The case where \( \lambda = 0 \) corresponds to the pole, and the other case corresponds to the point

\[
(x, y, z) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right).
\]

This gives us a parametrization for the unit 2-sphere \( S^2 \) excluding the pole, namely

\[
x(u, v) = \frac{2u}{1 + u^2 + v^2},
\]

\[
y(u, v) = \frac{2v}{1 + u^2 + v^2},
\]

\[
z(u, v) = \frac{1 - u^2 - v^2}{1 + u^2 + v^2}.
\]
for every \( u, v \in \mathbb{F} \) such that \( u^2 + v^2 \neq -1 \).

Similar to what we have done, the parametrization for the unit 3-sphere excluding the pole, by choosing the pole to be \((0, 0, 0, -1)\) would be

\[
\begin{align*}
\tau(u, v, w) &= \frac{2u}{1 + u^2 + v^2 + w^2}, \\
x(u, v, w) &= \frac{2v}{1 + u^2 + v^2 + w^2}, \\
y(u, v, w) &= \frac{2w}{1 + u^2 + v^2 + w^2}, \\
z(u, v, w) &= \frac{1 - u^2 - v^2 - w^2}{1 + u^2 + v^2 + w^2},
\end{align*}
\]

where \( u, v, w \in \mathbb{F} \) and \( u^2 + v^2 + w^2 + 1 \neq 0 \). This gives us the bijection between \( \mathbb{P}^3 \) and \( S^3 \), where the projective space \( \mathbb{P}^3 \subset \mathbb{A}^4 \) is taken to be the set lines through \((u, v, w, 0)\) and \((0, 0, 0, -1)\).

### 2.3.2 The Standard Theory: When \( \mathbb{F} = \mathbb{R} \)

When the underlying field \( \mathbb{F} \) is real numbers, we can have the usual parametrization of a sphere by using trigonometric functions \( \cos \) and \( \sin \). Equip the field \( \mathbb{R} \) with the usual order so that \( \mathbb{R} \) becomes an ordered field. The equation of a circle in the Cartesian plane \( \mathbb{R}^2 \) centered at \((a, b)\) and radius \( r \geq 0 \) is given by \((x - a)^2 + (y - b)^2 = r^2\), and every point on the circle can be specified with the following parametrization:

\[
\begin{align*}
x(r, \theta) &= a + r \cos \theta, \\
y(r, \theta) &= b + r \sin \theta,
\end{align*}
\]

where \( \theta \) ranges from 0 to \( 2\pi \). We call \((a + r \cos \theta, b + r \sin \theta)\) as the **polar coordinate** of the point \((x, y)\). Here \( r \) is called the radial coordinate and \( \theta \) the angular coordinate. Obviously, the case \( r = 0 \) corresponds to a point \((a, b)\). The parametrization of the unit circle is then just a special case of this, with \( a = b = 0 \) and \( r = 1 \). Hence the parametrization of the unit circle with equation \( x^2 + y^2 = 1 \) is given by

\[
\begin{align*}
x(\theta) &= \cos \theta \quad \text{and} \quad y(\theta) = \sin \theta, \quad \text{where} \quad 0 \leq \theta < 2\pi.
\end{align*}
\]

The case where \( \theta = 0 \) corresponds to the point \((1,0)\), \( \theta = \frac{\pi}{2} \) corresponds to the point \((0,1)\), and so on. Combining this and the stereographic projections
developed in the Subsection 3.3.1, we can conclude that for each \( \theta \in [0, 2\pi) \), there is \( u \in \mathbb{R} \) such that

\[
\cos \theta = \frac{1 - u^2}{1 + u^2} \quad \text{and} \quad \sin \theta = \frac{2u}{1 + u^2}.
\]

In this case, \( u^2 + 1 \) is always positive, so \( x(u) \) and \( y(u) \) are always defined for every \( u \in \mathbb{R} \).

If we go one dimension higher to consider the parametrization of general 2-sphere, which is given by \((x - a)^2 + (y - b)^2 + (z - c)^2 = 1\), it is only natural to parametrize it in the following manners:

\[
x (r, \phi, \theta) = a + r \sin \phi \cos \theta, \\
y (r, \phi, \theta) = b + r \sin \phi \sin \theta, \\
z (r, \phi, \theta) = c + r \cos \phi,
\]

where \( \theta \in [0, 2\pi) \) and \( \phi \in [0, \pi] \). Here we have two angular coordinates, \( \phi \) and \( \theta \). Implicitly, if we combine the parametrization of \( S^2 \) from stereographic projections above, this means that given any \( \phi \in [0, \pi] \) and \( \theta \in [0, 2\pi) \), one can always find a pair \((u, v) \in \mathbb{R}^2 \) so that

\[
\sin \phi \cos \theta = \frac{2u}{1 + u^2 + v^2}, \\
\sin \phi \sin \theta = \frac{2v}{1 + u^2 + v^2}, \\
\cos \phi = \frac{1 - u^2 - v^2}{1 + u^2 + v^2}.
\]

Note that again, we have \( u^2 + v^2 + 1 > 0 \), so \( x(u, v) \), \( y(u, v) \), and \( z(u, v) \) are always defined for every \((u, v) \in \mathbb{R}^2 \).

Analogous to the previous cases, the parametrization of 3-sphere where the equation of the sphere is given by \((t - a)^2 + (x - b)^2 + (y - c)^2 + (z - d)^2 = r^2\) is

\[
t (r, \psi, \phi, \theta) = a + r \sin \psi \sin \phi \cos \theta, \\
x (r, \psi, \phi, \theta) = b + r \sin \psi \sin \phi \sin \theta, \\
y (r, \psi, \phi, \theta) = c + r \sin \psi \cos \phi, \\
z (r, \psi, \phi, \theta) = d + r \cos \psi,
\]

where \( \theta \in [0, 2\pi) \), \( \phi \in [0, \pi] \) and \( \psi \in [0, \pi] \).
Chapter 3

Theory of Harmonic Polynomials

The theory of harmonic functions presented here is the standard analytical approach that works well in any dimension over the field $\mathbb{R}$, but we present it in a more restricted algebraic way so that we can extend the theory to more general fields. It is presented here to give the sense of what the analytical point of view looks like. Since this is usually framed on the field of real numbers $\mathbb{R}$, we will follow that. However, since the purpose of this thesis is to offer an algebraic point of view, certain concepts that have possibilities to work in a more general algebraic setting will be distinguished whenever necessary. For instance, the usual direction is to define the norm $\|x\|$ of $x$. Here we avoid that and instead work with a more general function, namely quadrance.

Most of the theory here is developed from Axler [6]. The bilinear form defined in Section 4.2 is different from Axler’s approach to decompose the space of homogeneous polynomials, and it is introduced to give a more algebraic approach compared to Axler’s more analytical approach. In particular, the proofs of Lemma 20, Corollary 21, Theorem 22, Corollary 23 and 24 before arriving at the Homogeneous Polynomial Decomposition Theorem are our own version. Every example is also our own, and they are intended to help illustrate the concepts better.

3.1 Introduction

Let $\mathbb{A}^n$ be an affine space over a field $F$. Here, and throughout this chapter, we will assume that $n > 2$, unless stated otherwise. Since we are going to talk about polynomials and rational functions on $\mathbb{A}^n$, we shall adopt the notion of formal
derivatives, where the derivative is purely algebraic. That means if we have a variable \( x \), we set
\[
\frac{d}{dx} (x^n) \equiv n x^{n-1}
\]
and we can generalize that to the multivariable case. Recall the Laplacian operator
\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}
\] (3.1)
which acts on polynomial functions on \( \mathbb{A}^n \). Again, the second derivation here is taken to be the formal derivative. Denote by \( P (\mathbb{A}^n) \) the space of all polynomial functions on \( \mathbb{A}^n \).

**Definition 3** A polynomial function \( u \in P (\mathbb{A}^n) \) is harmonic precisely when \( \Delta u \equiv 0 \).

We work mostly with polynomial functions, but the classical theory on \( \mathbb{R} \) is more general and occasionally we will give examples of other harmonic functions in this case. We will sometimes denote \( \Delta \) by
\[
\Delta = D_1^2 + D_2^2 + \cdots + D_n^2,
\] (3.2)
where \( D_i^2 \) denotes the second partial derivative with respect to the \( i \)-th variable. Also, denote by \( x = (x_1, x_2, \ldots, x_n) \) the typical point in \( \mathbb{A}^n \) and define the quadrance of \( x \) by \( Q(x) = x_1^2 + x_2^2 + \cdots + x_n^2 \). We will give some examples of harmonic functions when \( \mathbb{F} = \mathbb{Q} \).

**Example 3** Constant polynomials, linear polynomials are all harmonic functions.

**Example 4** Let \( p(x_1, x_2, x_3) = x_1^2 + x_2^2 - 2x_3^2 \). Since
\[
\Delta p = \frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2}
\]
\[
= 2 + 2 - 4 = 0,
\]
it follows that \( p \) is harmonic.

**Example 5** Let \( p(x, y) = 2x^2 + 5y^2 \). Over the field \( \mathbb{Q} \), \( p \) is not harmonic since \( \Delta p = 4 + 10 = 14 \neq 0 \). But over \( \mathbb{F}_7 \), \( p \) is harmonic since \( \Delta p = 14 = 0 \).
The above examples are all polynomials which are rational functions. Here is an example, when the underlying field is taken to be the real numbers \( \mathbb{R} \), that incorporates transcendental functions \( \exp \) and \( \sin \).

**Example 6** The function \( \psi (x_1, x_2) = e^{x_1} \sin x_2 \) is harmonic since

\[
\Delta \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = e^{x_1} \sin x_2 - e^{x_1} \sin x_2 = 0.
\]

Throughout this chapter, we shall assume that the underlying field is \( \mathbb{R} \), unless stated otherwise. The next example will be important for the rest of this chapter.

**Example 7** For all \( n > 2 \), the function \( p(x) = Q(x)^{1 - \frac{1}{2n}} \) is harmonic. To see this, note that

\[
D_i^2 p = (2 - n) \left[ (x_1^2 + x_2^2 + \cdots + x_n^2)^{-\frac{1}{2n}} - nx_i^2 (x_1^2 + x_2^2 + \cdots + x_n^2)^{-\frac{1}{2n} - 1} \right] = (2 - n) \left[ Q(x)^{-\frac{1}{2n}} - nx_i^2 Q(x)^{-\frac{1}{2n} - 1} \right]
\]

for all \( i \in \{1, 2, \ldots, n\} \). Hence

\[
\Delta p = \sum_{i=1}^{n} D_i^2 p = \sum_{i=1}^{n} (2 - n) \left[ Q(x)^{-\frac{1}{2n}} - nx_i^2 Q(x)^{-\frac{1}{2n} - 1} \right] = (2 - n) \sum_{i=1}^{n} \left( Q(x)^{-\frac{1}{2n}} - nx_i^2 Q(x)^{-\frac{1}{2n} - 1} \right) = (2 - n) \left( nQ(x)^{-\frac{1}{2n}} - nQ(x)^{-\frac{1}{2n} - 1} \sum_{i=1}^{n} x_i^2 \right) = (2 - n) \left( nQ(x)^{-\frac{1}{2n}} - nQ(x)^{-\frac{1}{2n}} \right) = 0.
\]

**Remark 3** Note that in the example above, \( p(x) \) is a rational function if \( n \) is even and not rational, otherwise. We can still extract this to a more general algebraic setting if \( n \) is even, treating the derivative as formal derivative. In both cases, \( p(x) \) is an algebraic function.
Below we list several basic properties of any harmonic polynomials.

**Proposition 4** Let $u$ be a harmonic polynomial defined on $\mathbb{A}^n$.

1. If $y \in \mathbb{A}^n$, then the function $v(x) = u(x - y)$, the $y$-translate of $u$, defined on $\mathbb{A}^n + y = \{w + y : w \in \mathbb{A}^n\}$ is also harmonic. This property is called translation invariance.

2. If $0 \neq r \in \mathbb{F}$, then the $r$-dilate of $u$, defined by $u_r \equiv u(rx)$ on $\mathbb{A}^n$ is also harmonic. This property is called dilation invariance.

3. If $T : \mathbb{A}^n \to \mathbb{A}^n$ is an orthogonal linear map, then the rotation of $u$ under $T$, denoted by $v = u \circ T$, is also harmonic. This property is called rotation invariance.

**Proof.** The first proposition is trivial, since

$$D_i^2v = D_i^2u,$$

so harmonicity of $v$ follows from harmonicity of $u$. For the second proposition, note that

$$\Delta (u_r) = \Delta (ru(x)) = r^2 (\Delta u)_r$$
on $\mathbb{A}^n$. It can be inferred that $u$ is harmonic if and only if $u_r$ is harmonic. Now we prove the third proposition. Recall that $T$ is orthogonal if $Q(T(x)) = Q(x)$ for all $x \in \mathbb{A}^n$. We can also say that $T$ is orthogonal precisely when $T$ preserves the unit sphere. It is well-known that if $T$ is orthogonal, then the columns of the matrix corresponding to $T$ with respect to the standard basis on $\mathbb{A}^n$ form an orthonormal basis. Let $[a_{ij}]$ be the matrix of $T$ relative to the standard basis on $\mathbb{A}^n$. For each $k \in \{1, 2, \ldots, n\}$,

$$\frac{\partial}{\partial x_k} (u \circ T) = \sum_{i=1}^{n} a_{ik} \left( \frac{\partial u}{\partial x_i} \right) \circ T.$$
Differentiating once more and taking the sum over \( k \) yields
\[
\Delta (u \circ T) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( \sum_{i=1}^{n} a_{ik} \left( \frac{\partial u}{\partial x_i} \right) \circ T \right)
\]
\[
= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} \left( \frac{\partial}{\partial x_k} \left( \frac{\partial u}{\partial x_i} \right) \right) \circ T
\]
\[
= \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} a_{jk} a_{ik} \left( \frac{\partial^2 u}{\partial x_j \partial x_i} \right) \circ T
\]
\[
= \sum_{i=1}^{n} \left( \frac{\partial^2 u}{\partial x_i^2} \right) \circ T
\]
\[
= (\Delta u) \circ T.
\]
Thus \( u \) is harmonic if and only if \( u \circ T \) is harmonic.

### 3.2 Harmonic Polynomials on the Sphere

Set \( P(k) \) to be the subspace of \( P \) consisting of all polynomials of degree less than or equal to \( k \). Clearly
\[
P(0) \subset P(1) \subset P(2) \subset \cdots \subset P(k).
\]
If we let \( P_k(\mathbb{A}^n) \) to be the space of all homogeneous polynomials of degree \( k \) on \( \mathbb{A}^n \), then any polynomial is a finite linear combination of monomials \( x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \) where \( m_1 + m_2 + \cdots + m_n = k \). We have, for any \( k \in \mathbb{N} \),
\[
P_0 \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_k = P(k).
\]
Denote by \( S^{n-1} \) the set of unit vectors in \( \mathbb{A}^n \); that is, the set of all \( x \in \mathbb{A}^n \) such that \( Q(x) = 1 \). This set will be called the \((n-1)\)-sphere inside \( \mathbb{A}^n \). If \( p \) is any polynomial on \( \mathbb{A}^n \), then denote by \( p|_{S^{n-1}} \) its restriction to \( S^{n-1} \). When the context is clear, we will use \( p|_S \) instead of \( p|_{S^{n-1}} \). For example, if \( n = 4 \) and
\[
p(x) = x_1^3 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 \in P_3(\mathbb{A}^4),
\]
then \( p|_S(x) = x_1 \) since
\[
x_1^3 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 = x_1 (x_1^2 + x_2^2 + x_3^2 + x_4^2) = x_1.
\]
We also denote $H(\mathbb{A}^n)$ and $H_k(\mathbb{A}^n)$ by the space of all harmonic polynomials on $\mathbb{A}^n$ and the space of all homogeneous harmonic polynomials of degree $k$ on $\mathbb{A}^n$, respectively. It is also evident that $P_k(\mathbb{A}^n)$ is a vector space over $\mathbb{F}$, and $H_k(\mathbb{A}^n)$ is a subspace of $P_k(\mathbb{A}^n)$.

It is sometimes convenient to write a homogeneous monomial

$$p(x) = p(x_1, x_2, \ldots, x_n) = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

on $\mathbb{A}^n$ by the notion of multi-index $\alpha = (m_1, m_2, \ldots, m_n)$. We define $|\alpha|$ to be just the sum $m_1 + m_2 + \cdots + m_n$. In this case we write $p$ as $x^\alpha$ in short. Since any homogeneous polynomial of degree $k$ is just a linear combination of monomials of the form $x^\alpha$ where $|\alpha| = k$, then every homogeneous polynomial of degree $k$ can be written as

$$p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha. \quad (3.3)$$

Define the bilinear form in the vector space $P_k(\mathbb{A}^n)$ over $\mathbb{F}$

$$\langle \cdot, \cdot \rangle : P_k(\mathbb{A}^n) \times P_k(\mathbb{A}^n) \to \mathbb{F}$$

$$(p, q) \mapsto (p(D)) q(x)|_{x=0}, \quad (3.4)$$

where $p(D)$ is the differential operator

$$p(D) = \sum_{|\alpha|=k} c_\alpha D^\alpha$$

associated to (3.3), with

$$D^\alpha = D^{(m_1, \ldots, m_n)} = \frac{\partial^{m_1} \cdots \partial^{m_n}}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} = \frac{\partial^k}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}.$$ 

Hence (3.4) is the differential operator $p(D)$ applied to $q(x)$ and evaluated at $x = 0$. For example, if $k = 3$ and $p(x) = p(x, y, z) = 5x^3 + 7xy^2 - 3z^3$, then

$$p(D) = 5 \frac{\partial^3}{\partial x^3} + 7 \frac{\partial^3}{\partial x \partial y^2} - 3 \frac{\partial^3}{\partial z^3}.$$ 

We have the following Lemma, which in brief tells us that the bilinear form of two different monomials are always zero.
Section 3. Theory of Harmonic Polynomials

Lemma 1 If \( p \) and \( q \) are two monomials where \( p \neq q \), then

\[ \langle p, q \rangle = 0. \]

**Proof.** Let \( p(x) = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \) and \( q(x) = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} \). Observe that

\[ \langle p, q \rangle = \left( \frac{\partial^{m_1} \partial^{m_2} \cdots \partial^{m_n}}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}} \right) (x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}) \bigg|_{x=0}. \]

Now, if \( m_i > l_i \) for at least one value of \( i \), then the above expression will be automatically zero. If \( m_i < l_i \) for at least one value of \( i \), then there will still be a term \( x_i^{k} \) for some \( k \), hence by evaluating \( x_i \) at 0 we have the above expression to be zero as well. So if \( \langle p, q \rangle \neq 0 \) it must be the case that \( m_i = l_i \) for all \( i \in \{1, 2, \ldots , n \} \).

**Corollary 2** If \( p \) is defined as above then

\[ \langle p, p \rangle = \prod_{i=1}^{n} m_i!. \]

**Proof.** We have that

\[ \langle p, p \rangle = \left( \frac{\partial^{m_1} \partial^{m_2} \cdots \partial^{m_n}}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}} \right) (x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) \bigg|_{x=0} = (m_1 \times \cdots \times 1) \times (m_2 \times \cdots \times 1) \times \cdots \times (m_n \times \cdots \times 1) = \prod_{i=1}^{n} m_i!, \]

as desired. ■

Now we will see the relation between the Laplacian operator \( \Delta \) and the quadrance \( Q \) described by using the Hermitian form defined above. The following shows that the Laplacian operator \( \Delta \) and the multiplication by \( Q(x) \) operator are adjoint.

**Theorem 6 (Relation of Laplacian and Quadrance)** If \( p \) and \( q \) are two monomials then,

\[ \langle \Delta p, q \rangle = \langle p, Q(x) q \rangle . \]

**Proof.** By the linearity of the bilinear form, we can just assume that both \( p \) and \( q \) are monomials. Let \( p(x) = x^{\alpha} \) where \( \alpha = (m_1, m_2, \ldots , m_n) \), \( q(x) = x^{\beta} \) where \( \beta = (l_1, l_2, \ldots , l_n) \). From the previous lemma, we must conclude that \( |\alpha| = k \) and \( |\beta| = k - 2 \) for some \( k \geq 2 \), otherwise \( \langle \Delta p, q \rangle = 0 = \langle p, Q(x) q \rangle \).
We have that
\[
\Delta p = m_1 (m_1 - 1) x_1^{m_1-2} \ldots x_n^{m_n} + \cdots + m_n (m_n - 1) x_1^{m_1} \ldots x_n^{m_n-2}
\]
so
\[
\langle \Delta p, q \rangle = \sum_{i=1}^{n} m_i (m_i - 1) \left( \frac{\partial^{m_1} \ldots \partial^{m_i-2} \ldots \partial^{m_n}}{\partial x_1^{m_1} \ldots \partial x_i^{m_i-2} \ldots \partial x_n^{m_n}} \right) (x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n}) \bigg|_{x=0}.
\]
(3.5)

Since \(|\alpha| - |\beta| = 2\), we have two cases. Either exactly \(m_i - l_i = 2\) for only one \(i\) and \(m_j - l_j = 1\) otherwise; or \(m_i - l_i = 1\) for exactly two values of \(i\) and \(m_j - l_j = 1\) otherwise. In the first case we can reduce the above equation to get
\[
\langle \Delta p, q \rangle = m_i (m_i - 1) (m_i! \ldots (m_i - 2)! \ldots m_n!) = \prod_{i=1}^{n} m_i!.
\]

In the second case, (3.5) will reduce to \(\langle \Delta p, q \rangle = 0\) since for two such values of \(i\), the derivative is taken \(m_i - 2\) times while the power of \(x_i\) is \(l_i = m_i - 1 > m_i - 2\).

Hence \(\langle \Delta p, q \rangle = m_1! \ldots m_n!\) precisely when there is exactly one \(i \in \{1, 2, \ldots, n\}\) such that \(m_i - l_i = 2\) and \(m_j = l_j\) for \(j \neq i\); zero otherwise.

Now, note that
\[
\langle p, Q (x) q \rangle = \left( \frac{\partial^{m_1} \partial^{m_2} \ldots \partial^{m_n}}{\partial x_1^{m_1} \partial x_2^{m_2} \ldots \partial x_n^{m_n}} \right) (x_1^{l_1+2} x_2^{l_2} \ldots x_n^{l_n} + \cdots + x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n+2}) \bigg|_{x=0}.
\]
Again, if exactly \(m_i - l_i = 2\) for only one \(i\) and \(m_j = l_j\) otherwise, then the above equation reduces to
\[
\langle p, Q (x) q \rangle = \left( \frac{\partial^{m_1} \partial^{m_2} \ldots \partial^{m_n}}{\partial x_1^{m_1} \partial x_2^{m_2} \ldots \partial x_n^{m_n}} \right) (x_1^{l_1} \ldots x_i^{l_i+2} \ldots x_n^{l_n}) \bigg|_{x=0} = l_1! \ldots (l_i + 2)! \ldots l_n! = m_1 \ldots m_i! \ldots m_n! = \prod_{i=1}^{n} m_i!.
\]

If exactly \(m_i - l_i = 1\) for two values of \(i\) and \(m_j = l_j\) otherwise, by the same reasoning, \(\langle p, Q (x) q \rangle = 0\). Since \(\langle \Delta p, q \rangle\) and \(\langle p, Q (x) q \rangle\) agree on every case, we conclude that \(\langle \Delta p, q \rangle = \langle p, Q (x) q \rangle\). 

Relation of Laplacian and Quadrance Theorem above gives a couple of interesting corollaries.
Corollary 3 For $k \geq 2$, the Laplacian map

$$\Delta : P_k(\mathbb{A}^n) \mapsto P_{k-2}(\mathbb{A}^n)$$

is surjective.

Proof. For all $p \in P_k(\mathbb{A}^n)$ such that $\langle \Delta p, q \rangle = 0$, we have that

$$0 = \langle \Delta p, q \rangle = \langle p, Q(x)q \rangle.$$

More specifically, if we pick $p = Q(x)q$, we have that $\langle Q(x)q, Q(x)q \rangle = 0$ which implies that $Q(x)q = 0$, for if $Q(x)q \neq 0$, then by Corollary 2, $\langle Q(x)q, Q(x)q \rangle \neq 0$. Since this is true for all $x$, we must have that $q = 0$. Hence $\Delta$ maps $P_k(\mathbb{A}^n)$ onto $P_{k-2}(\mathbb{A}^n)$. ■

Corollary 4 With respect to the Hermitian form (3.4), all harmonic polynomials $f \in H_k(\mathbb{A}^n)$ are orthogonal to $Q(x)g$ for $g \in P_{k-2}(\mathbb{A}^n)$.

Proof. Take any $f \in H_k(\mathbb{A}^n)$, then for any $g \in P_{k-2}(\mathbb{A}^n)$ we have

$$\langle f, Q(x)g \rangle = \langle \Delta f, g \rangle = 0,$$

where the second equality is justified since $\Delta f = 0$. ■

Corollary 4 above is an important ingredient for the following theorem that is the heart of this section:

Theorem 7 (Homogeneous Polynomials Decomposition) If $k \geq 2$, then

$$P_k(\mathbb{A}^n) = H_k(\mathbb{A}^n) \oplus Q(x)P_{k-2}(\mathbb{A}^n).$$

The above theorem is the key to the following theorem which states explicitly the decomposition of any homogeneous polynomial of degree $k$.

Theorem 8 Any polynomial $p(x) = p(x_1, x_2, \ldots, x_n) \in P_k(\mathbb{A}^n)$ can be uniquely written in the form

$$p = p_k + Q(x)p_{k-2} + \cdots + Q(x)^m p_{k-2m},$$

(3.6)

where $m = \left\lfloor \frac{k}{2} \right\rfloor$ and each $p_i \in H_i(\mathbb{A}^n)$.  

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Proof. The result is obvious for \( k = 0 \) or \( k = 1 \), since any homogeneous polynomial of degree 0 or 1 is harmonic. Suppose then \( k \geq 2 \). By the Homogeneous Polynomial Decomposition Theorem, there is a unique decomposition

\[
p = p_k + Q(x) q_{k-2}
\]

where \( p_k \in H_k (\mathbb{A}^n) \), and \( q_{k-2} \in P_{k-2} (\mathbb{A}^n) \). We can proceed by taking the unique decomposition for \( q_{k-2} \) as

\[
q_{k-2} = p_{k-2} + Q(x) q_{k-4}
\]

for \( p_{k-2} \in H_{k-2} (\mathbb{A}^n) \), and \( q_{k-4} \in P_{k-4} (\mathbb{A}^n) \) so that

\[
p = p_k + Q(x) (p_{k-2} + Q(x) q_{k-4})
= p_k + Q(x) p_{k-2} + Q(x)^2 q_{k-4}.
\]

Continuing in this fashion, after finitely many steps we will arrive at (3.6).

As a corollary of the Theorem 8, if we have any polynomial \( p \in P_k (\mathbb{A}^n) \) as above, then

\[
p|_S = p_k|_S + p_{k-2}|_S + \cdots + p_{k-2m}|_S
\]

where \( m \) and \( p_i \) are defined also as above.

Now, we turn our attention back to the Homogeneous Polynomials Decomposition Theorem above, and restrict everything to the unit sphere. To be more precise, we have that

\[
P_k (\mathbb{A}^n) = H_k (\mathbb{A}^n) \oplus Q(x) P_{k-2} (\mathbb{A}^n)
\]

from the theorem. If we just restrict everything in \( S^{n-1} \subset \mathbb{A}^n \), then the left hand side becomes \( P_k (S^{n-1}) \) while the right hand side becomes \( H_k (S^{n-1}) \oplus P_{k-2} (S^{n-1}) \). We can further decompose \( P_{k-2} (S^{n-1}) \) as \( H_{k-2} (S^{n-1}) \oplus P_{k-4} (S^{n-1}) \) until we are left with \( P_{k-2m} (S^{n-1}) \) with \( m = \lfloor \frac{k}{2} \rfloor \). However note that for this \( m \), we have \( P_{k-2m} (S^{n-1}) = H_{k-2m} (S^{n-1}) \) since \( k - 2m \) is either 0 or 1, depending on the parity of \( k \). Hence we have

\[
P_k (S^{n-1}) = H_k (S^{n-1}) \oplus H_{k-2} (S^{n-1}) \oplus \cdots \oplus H_{k-2m} (S^{n-1})
= \bigoplus_{i=0}^{m} H_{k-2i} (S^{n-1})\,.
\]
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The above relation shows that we can regard $P_k (S^{n-1})$ as the finite direct sum of the spaces $H_{k-2i} (S^{n-1})$. Moreover, the spaces $H_i (S^{n-1})$ are pairwise orthogonal with respect to the inner product

$$
\langle p, q \rangle = \int_{S^{n-1}} pq \, d\mu,
$$

where $\mu$ is a surface measure on $S^{n-1}$. The following theorem uses the analytic approach so the underlying field is $\mathbb{R}$.

**Theorem 9** If $p$ and $q$ are polynomials on $\mathbb{R}^n$ and $q$ is harmonic and homogeneous with degree higher than the degree of $p$, then

$$
\int_{S^{n-1}} pq \, d\mu = 0.
$$

**Proof.** By linearity and the fact that homogeneous polynomials restricted to a sphere can be decomposed as a sum of harmonic polynomials, without loss of generality, we can assume that $p \in H_k (\mathbb{R}^n)$ and $q \in H_m (\mathbb{R}^n)$, where $k < m$. Recall Green’s identity

$$
\int_{B^{n-1}} (u \Delta v - v \Delta u) \, dV = \int_{S^{n-1}} (u \partial_{n} v - v \partial_{n} u) \, ds,
$$

where $B^{n-1}$ is the unit open ball centered at the origin, $u$ and $v$ are both twice-differentiable functions defined on a neighborhood of $\overline{B}$, the closure of $B^{n-1}$, $dV$ and $d\mu$ are Lebesgue volume measure defined on $\mathbb{R}^n$ and surface area measure on $S^{n-1}$, respectively. By Green’s theorem, we have

$$
\int_{S^{n-1}} (p \partial_{n} q - q \partial_{n} p) \, d\mu = 0.
$$

Now, for any $\xi \in S^{n-1}$,

$$(D_{n} p) (\xi) = \frac{d}{dr} p (r \xi) \bigg|_{r=1} = \frac{d}{dr} (r^k p (\xi)) \bigg|_{r=1} = kp (\xi),$$

and by similar reasoning, $(D_{n} q) (\xi) = mq$ on $S^{n-1}$, so

$$0 = \int_{S^{n-1}} (p \partial_{n} q - q \partial_{n} p) \, d\mu = \int_{S^{n-1}} (mpq - kpq) \, d\mu = (m - k) \int_{S^{n-1}} pq \, d\mu.$$
Since $m > k$, we must have
\[
\int_{S^{n-1}} pq \, d\mu = 0,
\]
as desired. \[\Box\]

Theorem 9 says that a polynomial $p$ of degree $m$ will be orthogonal to another polynomial $q$ of higher degree as long as $q$ is harmonic. As an important special case, we have the following Theorem.

**Theorem 10** Two homogeneous harmonic polynomials of different degree are orthogonal with respect to the inner product (3.8), so that $H_k(S^{n-1})$ and $H_m(S^{n-1})$ are orthogonal as long as $k \neq m$.

To close this section, we will give an explicit formula for $\dim H_k(\mathcal{A}^n)$. Note that if $k = 0$ then $H_0(\mathcal{A}^n)$ is just the set of constant polynomials, so $\dim H_0(\mathcal{A}^n) = 1$. If $k = 1$, then $H_1(\mathcal{A}^n)$ is just the space of all linear polynomials on $\mathcal{A}^n$, so $\dim H_1(\mathcal{A}^n) = n$.

**Theorem 11 (Dimension of $H_k(\mathcal{A}^n)$)** If $k \geq 2$, then
\[
\dim H_k(\mathcal{A}^n) = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}.
\] (3.10)

**Proof.** First we begin by finding $\dim P_k(\mathcal{A}^n)$. We know that the set of monomials $\{x^\alpha : \alpha = (m_1, m_2, \ldots, m_n)\}$ and $|\alpha| = k$ is a basis for $P_k(\mathcal{A}^n)$. Hence, the dimension of $P_k(\mathcal{A}^n)$ must be equal to the number of distinct multi-indices $\alpha$ such that $m_1 + m_2 + \cdots + m_n = k$ for each $m_i \geq 0$. A basic combinatorial argument (see Epp [14], for instance) tells us that the number of such solutions is
\[
\binom{n+k-1}{n-1}.
\]
Hence, from the Homogeneous Polynomials Decomposition Theorem, we deduce that
\[
\dim H_k(\mathcal{A}^n) = \dim P_k(\mathcal{A}^n) - \dim P_{k-2}(\mathcal{A}^n)
= \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1},
\]
as desired. \[\Box\]
As a particular example, if \( n = 4 \), we have
\[
\dim H_k (\mathbb{A}^4) = \binom{k + 3}{3} - \binom{k + 1}{3} = (k + 1)^2.
\] (3.11)

Now, one may wonder, given any homogeneous polynomial of degree \( k \), can we find its decomposition represented in (3.6)? If any, is there a finite simple algorithm to find such a decomposition? Although the fact affirms the statements proposed by both questions, we need several basic concepts in hand first.

### 3.3 The Kelvin Transform

#### 3.3.1 Inversion in the Unit Sphere

When studying functions on unbounded open sets in \( \mathbb{A}^n \), it is often convenient to append a point at infinity to \( \mathbb{A}^n \). From the usual stereographic projection, \( \mathbb{A}^{n-1} \cup \{ \infty \} \) is homeomorphic to the unit sphere \( S^{n-1} \) in \( \mathbb{A}^n \). Suppose that \( Q (x) = 0 \) precisely when \( x = 0 \). This works for \( \mathbb{Q} \) and real extensions of \( \mathbb{Q} \).

Consider the map \( x \mapsto x^* \) where
\[
x^* = \begin{cases} 
\frac{x}{Q(x)} & \text{if } x \neq 0, \infty \\
0 & \text{if } x = \infty \\
\infty & \text{if } x = 0
\end{cases}.
\]

This map is called the inversion of \( \mathbb{A}^n \) relative to the unit sphere \( S^{n-1} \). Note that if \( x \neq 0 \) or \( \infty \), then \( x^* \) lies on the line through the origin determined by \( x \), with
\[
Q (x^*) = Q \left( \frac{x}{Q(x)} \right) = \frac{1}{Q (x)^2} Q (x) = Q (x)^{-1}.
\]

#### 3.3.2 Maps Preserving Harmonic Functions

The inversion function preserves harmonic functions when \( n = 2 \), as given in the following proposition.

**Proposition 5** Let \( f \) be any harmonic polynomial on some set \( \Omega \subset \mathbb{A}^2 \setminus \{0\} \). Then the function \( g : x \mapsto f (x^*) \) is harmonic on \( \Omega^* = \{ x^* : x \in \Omega \} \).

**Proof.** Let \( f = f (x, y) \) be a harmonic polynomial on \( \Omega \), then
\[
g(x, y) = f \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).
\]
Now let
\[ u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{y}{x^2 + y^2} \]
so that
\[ \frac{\partial^2 g}{\partial x^2} = \frac{\partial f}{\partial u} \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 u} \left( \frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 f}{\partial u \partial v} \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) 
+ \frac{\partial f}{\partial v} \frac{\partial^2 v}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} \left( \frac{\partial v}{\partial x} \right)^2. \]
Similarly
\[ \frac{\partial^2 g}{\partial y^2} = \frac{\partial f}{\partial u} \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 u} \left( \frac{\partial u}{\partial y} \right)^2 + 2 \frac{\partial^2 f}{\partial u \partial v} \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial y} \right) 
+ \frac{\partial f}{\partial v} \frac{\partial^2 v}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} \left( \frac{\partial v}{\partial y} \right)^2. \]
Using the fact that
\[ \frac{\partial u}{\partial x} = \frac{-x^2 - y^2}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = \frac{\partial v}{\partial x}, \]
one may deduce that
\[ \Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} 
= \frac{\partial f}{\partial u} \left( \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} \right) + \frac{\partial^2 f}{\partial^2 u} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right) 
+ 2 \frac{\partial^2 f}{\partial u \partial v} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) + \frac{\partial f}{\partial v} \left( \frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} \right) 
+ \frac{\partial^2 f}{\partial^2 v} \left( \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) 
= 0. \]
Hence \( g \) is harmonic.

However, the inversion map does not preserve harmonicity for \( n > 2 \). One example is \( u(x) = Q(x)^{1-\frac{1}{2n}} \). This is a rational function when \( n \) is even. While \( u \) is harmonic, we see that
\[ r(x) = u(x^*) = Q \left( \frac{x}{Q(x)} \right)^{1-\frac{1}{2n}} = Q(x)^{\frac{1}{2n-1}} \]
which is not in general harmonic. Fortunately there is indeed a transformation which preserves harmonicity for \( n > 2 \), called the Kelvin transformation.
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**Definition 4** Given a polynomial \( u \) defined on \( E = \mathbb{A}^n \setminus \{0\} \), define the Kelvin transform \( K(u) \) on \( E^* \) by

\[
K(u)(x) = Q(x)^{1 - \frac{1}{2^n}} u(x^*).
\]

**Remark 4** Again, note that the transformation above is gives rational function precisely when \( n \) is even. Note also that \( K \) is a linear function.

One major property of this transformation is that it is its own inverse, as we will see in the following proposition.

**Proposition 6** If \( u \) is a polynomial defined on \( E = \mathbb{A}^n \setminus \{0\} \), then

\[
K(K(u)) = u.
\]

**Proof.** Note that

\[
K(K(u)(x)) = K(Q(x)^{1 - \frac{1}{2^n}} u(x^*))
= Q(x)^{1 - \frac{1}{2^n}} (Q(x^*)^{1 - \frac{1}{2^n}} u((x^*)^*))
= Q(x)^{1 - \frac{1}{2^n}} Q(x)^{\frac{1}{2^n} - 1} u(x)
= u(x)
\]

for every \( x \in E \). Hence \( K(K(u)) = u \). 

The Kelvin transform of any harmonic function is harmonic. To see this, we observe the following Lemma.

**Lemma 2** If \( p \) is a homogeneous polynomial of degree \( k \) on \( \mathbb{A}^n \), then

\[
\Delta \left( Q(x)^{1 - \frac{1}{2^n} k} p \right) = Q(x)^{1 - \frac{1}{2^n} k} \Delta p.
\]

**Proof.** The proof makes use of the product rule for Laplacians, which is given by

\[
\Delta (uv) = u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u
\]

for any twice continuously differentiable functions \( u \) and \( v \). By letting \( u = Q(x)^{1 - \frac{1}{2^n} k} \) and \( v = p \), we get

\[
\Delta \left( Q(x)^{1 - \frac{1}{2^n} k} p \right) = Q(x)^{1 - \frac{1}{2^n} k} \Delta p + 2\nabla \left( Q(x)^{1 - \frac{1}{2^n} k} \right) \cdot \nabla p + p\Delta \left( Q(x)^{1 - \frac{1}{2^n} k} \right)
\]

\[
= Q(x)^{1 - \frac{1}{2^n} k} \Delta p + 2(2 - n - 2k) Q(x)^{-\frac{1}{2^n} k} (x \cdot \nabla p)
+ p(2 - n - 2k)(-2k) Q(x)^{-\frac{1}{2^n} k}
\]

\[
= Q(x)^{1 - \frac{1}{2^n} k} \Delta p + 2(2 - n - 2k) Q(x)^{-\frac{1}{2^n} k} (x \cdot \nabla p - kp)
\]

\[
= Q(x)^{1 - \frac{1}{2^n} k} \Delta p,
\]
since for any homogeneous polynomial \( p = \sum_{|\alpha|=k} P_\alpha x^\alpha \) with degree \( k \), we have that

\[ x \cdot \nabla p = kp. \]

To see that, note that if \( p \) is a monomial \( x^\alpha \) where \( \alpha = (m_1, m_2, \ldots, m_n) \) and \( |\alpha| = k \), we must have that

\[
\begin{array}{c}
\mathbf{x} \cdot \nabla p = \\
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{pmatrix} \cdot \\
\begin{pmatrix}
m_1 x_1^{m_1-1} x_2^{m_2} \cdots x_n^{m_n} \\
m_2 x_1^{m_1} x_2^{m_2-1} \cdots x_n^{m_n} \\
\vdots \\
m_n x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n-1} \\
\end{pmatrix} \\
= (m_1 + m_2 + \cdots + m_n) x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \\
= kp.
\end{array}
\]

Hence if \( p \) is any homogeneous polynomial \( p = \sum_{|\alpha|=k} P_\alpha x^\alpha \) with degree \( k \), the result follows from the linearity of the dot product.

**Remark 5** In Lemma 2 above, we use the analytic approach again to find the derivative of \( u \) since \( u \) might not be a rational function if \( n \) is odd. We do not have to use analytic approach if \( n \) is even.

Now we have come to the main theorem about the Kelvin transform.

**Theorem 12 (Preservation of Harmonic Polynomials)** If \( p \) is any polynomial in \( \mathbb{A}^n \setminus \{0\} \), then \( p \) is harmonic if and only if \( K(p) \) is harmonic.

**Proof.** If \( p \) is harmonic (i.e. \( \Delta p = 0 \)), then \( \Delta (K(p)(x)) = K(Q(x)^2 \Delta p(x)) = 0 \). It then follows that \( K(p) \) is harmonic. Now we prove the other direction. First suppose \( p \) is a homogeneous polynomial of degree \( k \) on \( \mathbb{A}^n \setminus \{0\} \). Then observe that

\[
K(p)(x) = Q(x)^{1-\frac{k}{n}} p(x^*) \\
= Q(x)^{1-\frac{k}{n}} p \left( \frac{x}{Q(x)} \right) \\
= Q(x)^{1-\frac{k}{n}} \frac{1}{Q(x)^k} p(x) \\
= Q(x)^{1-\frac{k}{n}-k} p(x). \quad (3.12)
\]
With this in mind, we note that
\[
\Delta (K (p) (x)) = \Delta \left( Q (x)^{1 - \frac{1}{2} n - k} p(x) \right)
\]
\[
= Q (x)^{1 - \frac{1}{2} n - k} \Delta p(x)
\]
\[
= K (Q (x)^2 \Delta p(x)).
\] (3.13)

The first equality comes from the previous observation. For the second equality, note that \( Q (x)^2 \Delta p(x) \) is a homogeneous polynomial of degree \( k + 2 \), hence by applying the reasoning in the previous observation at Equation (3.12), replacing \( \pi (x) \) with \( Q (x)^2 \Delta \pi (x) \), we find that

\[
K (Q (x)^2 \Delta p(x)) = Q (x)^{1 - \frac{1}{2} n - (k + 2)} Q (x)^2 \Delta p(x)
\]
\[
= Q (x)^{1 - \frac{1}{2} n - k} \Delta p(x)
\]

which justifies the second equality in Equation (3.13).

By this and the linearity of \( K \), we conclude that the result is true for any polynomial. If we have \( \Delta K (p) (x) \equiv 0 \) (i.e. if \( K (p) \) is harmonic) then it implies that \( K (Q (x)^2 \Delta p(x)) \equiv 0 \). Applying the Kelvin transform to both sides of the last equality and using the fact that the Kelvin transform is its own inverse, we may deduce that

\[
Q (x)^2 \Delta p(x) \equiv 0.
\]

Since we have excluded 0 from our domain, then \( Q (x) \neq 0 \), which implies that \( \Delta p = 0 \), proving that \( p \) is harmonic. ■

### 3.4 Spherical Harmonics via Differentiation

Recall that for any polynomial \( p(x) = \sum \alpha c_{\alpha} x^{\alpha} \) on \( \mathbb{A}^n \), we can associate to it the differential operator \( p(D) = \sum \alpha c_{\alpha} D^{\alpha} \). The following lemma will be useful to understand the decomposition of homogeneous polynomials.

**Lemma 3** Suppose \( n > 2 \). If \( p \in P_k (\mathbb{A}^n) \), then \( K \left( p(D) Q (x)^{1 - \frac{k}{2}} \right) \in H_k (\mathbb{A}^n) \).

**Proof.** We will show first that \( K \left( p(D) Q (x)^{1 - \frac{k}{2}} \right) \) is homogeneous of degree \( k \). First we show that the statement holds when \( p \) is a monomial by induction. The statement is true when \( k = 0 \), since if \( k = 0 \) then \( p \) will just be a constant
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polynomial, say \( p(x) = c \). Then

\[
K \left( p(D) Q(x)^{1-\frac{1}{2^n}} \right) = K \left( c Q(x)^{1-\frac{1}{2^n}} \right) \\
= c K \left( Q(x)^{1-\frac{1}{2^n}} \right) \\
= c Q(x)^{1-\frac{1}{2^n}} Q(x^*)^{1-\frac{1}{2^n}} \\
= c,
\]

which is homogeneous of degree \( k = 0 \). Now for the inductive part, assume that the proposition is true for some fixed \( k \). Let \( \alpha \) be a multi-index with \( |\alpha| = k \), and our polynomial is \( p(x) = x^\alpha \). This implies that \( p(D) = D^\alpha \) and by our induction hypothesis, there exists \( u \in H_k(\mathbb{A}^n) \) such that

\[
K \left( D^\alpha Q(x)^{1-\frac{1}{2^n}} \right) = u(x).
\]

If we take the Kelvin transform of both sides, we get

\[
D^\alpha Q(x)^{1-\frac{1}{2^n}} = Q(x)^{1-\frac{1}{2^n}}u(x^*) \\
= Q(x)^{1-\frac{1}{2^n}-k}u(x).
\]

Fix an index \( i \in \{1, 2, \ldots, n\} \) and differentiate both sides with respect to \( x_i \). We have that

\[
\frac{\partial}{\partial x_i} D^\alpha Q(x)^{1-\frac{1}{2^n}} = (2 - n - 2k)x_i Q(x)^{-\frac{1}{2^n}-k}u(x) + Q(x)^{1-\frac{1}{2^n}-k} \frac{\partial u}{\partial x_i} \\
= Q(x)^{1-\frac{1}{2^n}-(k+1)} \left( (2 - n - 2k)x_i u(x) + Q(x) \frac{\partial u}{\partial x_i} \right) \\
= Q(x)^{1-\frac{1}{2^n}-(k+1)} v(x) \tag{3.14}
\]

where

\[
v(x) = (2 - n - 2k)x_i u(x) + Q(x) \frac{\partial u}{\partial x_i} \in P_{k+1}(\mathbb{A}^n).
\]

If we take the Kelvin transform of both sides of the Equation (3.14), we get

\[
K \left( \frac{\partial}{\partial x_i} D^\alpha Q(x)^{1-\frac{1}{2^n}} \right) = v(x).
\]

Since \( \frac{\partial}{\partial x_i} D^\alpha Q(x)^{1-\frac{1}{2^n}} \) represents differentiation with any arbitrary multi-index of size \( k+1 \), this completes the inductive argument. The result for any homogeneous polynomial \( p \) follows from the linearity of \( K \).
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We still need to prove that $K \left( p(D) Q (x)^{1 - \frac{1}{2n}} \right)$ is harmonic. Note that $Q (x)^{1 - \frac{1}{2n}}$ is harmonic and every partial derivative of a harmonic function is harmonic, hence $p(D) Q (x)^{1 - \frac{1}{2n}}$ is harmonic. Since the Kelvin transform of any harmonic function is harmonic, this completes the proof of the Lemma.

Recall that given any $\varphi \in \mathbb{A}^n$, we have a unique decomposition $\varphi = \varphi_{\mathbb{A}^n} + \varphi_{\mathbb{A}^n - 2}$, where $\varphi_{\mathbb{A}^n} \in \mathbb{A}^n_{\mathbb{A}^n}$ and $\varphi_{\mathbb{A}^n - 2} \in \mathbb{A}^n_{\mathbb{A}^n - 2}$. Combined with the Lemma 3, given any $p \in P_k (\mathbb{A}^n)$, this polynomial corresponds with two homogeneous harmonic polynomials of degree $k$, namely $p_k$ and $K \left( p(D) Q (x)^{1 - \frac{1}{2n}} \right)$. As we will see in the next important theorem, these two polynomials are related to each other, in the sense that they are multiples of each other.

**Theorem 13** If $n > 2$ and $p \in P_k (\mathbb{A}^n)$, then

$$K \left( p(D) Q (x)^{1 - \frac{1}{2n}} \right) = c_{k,n} \left( p - Q (x) q_{k-2} \right)$$

for some $q_{k-2} \in P_{k-2} (\mathbb{A}^n)$ and constant $c_{k,n}$ which is dependent on $k$ and $n$ defined by

$$c_{k,n} = \prod_{m=0}^{k-1} (2 - n - 2m).$$

By convention, since for $k = 0$ and $k = 1$ any polynomial is harmonic, then $p = p_k$ and hence $q = 0$.

**Proof.** The proof is somewhat similar to the proof in the Lemma 3. First, a comment about the constant $c_{k,n}$. In the proof of the Lemma 3, the extra constant $(2 - n - 2k)$ appears when we move from any monomial of degree $k$ to $k + 1$. Consider the case where $p$ is a monomial, since we can prove the general result when $p$ is any homogeneous polynomial by linearity of $K$. The result is true for $k = 0$, since we can choose $q = 0$ and $c_{m,n} = 1$. Suppose that the result is true for some fixed value $k$. Let $\alpha$ be a multi-index with size $k$, and $p(x) = x^\alpha$. By our induction hypothesis, there exists $q_{k-2} \in P_{k-2} (\mathbb{A}^n)$ such that

$$K \left( D^\alpha Q (x)^{1 - \frac{1}{2n}} \right) = c_{k,n} (x^\alpha - Q (x) q_{k-2}).$$

Set $u(x) = c_{k,n} (x^\alpha - Q (x) q_{k-2})$ and note that $u$ is a homogeneous harmonic polynomial of degree $k$. Applying the Kelvin transform to both sides of the
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We then have

\[ D^\alpha Q(x)^{1-\frac{k}{n}} = Q(x)^{1-\frac{k}{n}-k} u(x). \]

Fix an index \( i \) and differentiate both sides with respect to \( x_i \). We then have, as before,

\[
\frac{\partial}{\partial x_i} D^\alpha (x)^{1-\frac{k}{n}} = Q(x)^{1-\frac{k}{n}-(k+1)} c_{k,n} (2-n-2k) x_i (x^\alpha - Q(x) q_{k-2}) + \frac{1}{c_{k,n}} Q(x) \frac{\partial u}{\partial x_i},
\]

for some \( v \in P_{k-1}(\mathbb{A}^n) \). Now take the Kelvin transform of both sides of the equation above to get

\[
K \left( \frac{\partial}{\partial x_i} D^\alpha Q(x)^{1-\frac{k}{n}} \right) = c_{k+1,n} (x_i x^\alpha - Q(x) v).
\]

Because \( x_i x^\alpha \) represents arbitrary monomial of degree \( k+1 \), this completes the inductive procedure.

In the next theorem we will be able to say something about the algorithm to find the unique decomposition of any homogeneous polynomial mentioned above.

**Theorem 14 (Canonical Projection)** Suppose \( n > 2 \) and \( p \in P_k(\mathbb{A}^n) \). Then the canonical projection of \( p \) onto \( H_k(\mathbb{A}^n) \) is

\[
K \left( p(D) Q(x)^{1-\frac{k}{n}} \right) \frac{c_{k,n}}{c_{k,n}}.
\]

**Proof.** From Theorem 13, we can write

\[
p = K \left( p(D) Q(x)^{1-\frac{k}{n}} \right) \frac{c_{k,n}}{c_{k,n}} + Q(x) q_{k-2}.
\]

We know that the first term on the right hand side is harmonic, hence this is the unique decomposition of \( p \) mentioned above, so that \( K \left( p(D) Q(x)^{1-\frac{k}{n}} \right) \frac{c_{k,n}}{c_{k,n}} \) is the canonical projection of \( p \) onto \( H_k(\mathbb{A}^n) \). ■

Now let \( n = 4 \) and we will shortly see how we can decompose any given homogeneous polynomial of degree \( k \). Recall from Equation (3.6) that for \( p \in P_k(\mathbb{A}^4) \), there is a unique decomposition of the form

\[
p = p_k + (t^2 + x^2 + y^2 + z^2) p_{k-2} + \cdots + (t^2 + x^2 + y^2 + z^2)^m p_{k-2m},
\]
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where \( m = \left[ \frac{k}{2} \right] \) and \( p_i \in H_i (\mathbb{A}^4) \). From the Canonical Projection Theorem, we can compute \( p_i \) by using this simple algorithm. First, given any \( p \in P_k (\mathbb{A}^4) \), we can compute \( p_k \) explicitly by using the previous theorem. Once we find \( p_k \), we move to the second step, that is to find \( q_{k-2} \in P_{k-2} (\mathbb{A}^4) \) in the decomposition \( p = p_k + Q (x) q_{k-2} \). We then find \( p_{k-2} \) by using the same procedure described above, with \( q_{k-2} \) in place of \( p \) this time. After iterating the process \( m \) more times, we will arrive at \( p_{k-2m} \) and thus complete the decomposition process.

Let us observe the decomposition process more closely. We now give some examples that illustrate this decomposition. These examples are original.

**Example 8** Consider \( p(t, x, y, z) = x^2 \in P_2 (\mathbb{A}^4) \). It is not harmonic, since \( \Delta p = 2 \neq 0 \). Since \( k = 2 \), we would expect to find two harmonic polynomials, namely \( p_2 \) and \( p_0 \). Based on Theorem 14,

\[
p_2 = \frac{K \left( p(D) Q(x)^{-1} \right)}{c_{2,4}}
\]

\[
= \frac{1}{8} K \left( \frac{\partial}{\partial x^2} \left( \frac{1}{t^2 + x^2 + y^2 + z^2} \right) \right)
\]

\[
= \frac{1}{8} K \left( -2 \left( t^2 + x^2 + y^2 + z^2 \right)^{-2} + 8x^2 \left( t^2 + x^2 + y^2 + z^2 \right)^{-3} \right)
\]

\[
= -\frac{1}{4} K (Q(x)^{-2}) + K \left( x^2 Q(x)^{-3} \right)
\]

\[
= -\frac{1}{4} Q(x)^{1-2} Q \left( \frac{x}{Q(x)} \right)^{-2} + Q(x)^{1-2} \left( \frac{x}{Q(x)} \right)^2 Q \left( \frac{x}{Q(x)} \right)^{-3}
\]

\[
= -\frac{1}{4} Q(x) + x^2
\]

\[
= -\frac{1}{4} \left( t^2 + x^2 + y^2 + z^2 \right) + x^2
\]

\[
= -\frac{1}{4} t^2 + \frac{3}{4} x^2 - \frac{1}{4} y^2 - \frac{1}{4} z^2.
\]

The next step is to find \( q_0 \). From the decomposition \( p = p_2 + Q(x) q_0 \), we see that

\[
x^2 = -\frac{1}{4} t^2 + \frac{3}{4} x^2 - \frac{1}{4} y^2 - \frac{1}{4} z^2 + \left( t^2 + x^2 + y^2 + z^2 \right) q_0,
\]

which means that \( q_0 = \frac{1}{4} \). From this, we can find \( p_0 \) using the algorithm above. But since \( q_0 \) is already harmonic, then \( p_0 = \frac{1}{4} \). Hence, the unique decomposition of \( p \) in this case is

\[
p(x) = x^2 = \left( -\frac{1}{4} t^2 + \frac{3}{4} x^2 - \frac{1}{4} y^2 - \frac{1}{4} z^2 \right) + \frac{1}{4} Q(x).
\]
Hence, on $S^3$,
\[ p(x) = -\frac{1}{4}t^2 + \frac{3}{4}x^2 - \frac{1}{4}y^2 - \frac{1}{4}z^2 + \frac{1}{4}. \]

**Example 9** Consider $p(t, x, y, z) = t^3z \in P_4(A^4)$. Since $k = 4$, we need to find three harmonic polynomials, namely $p_4$, $p_2$, and $p_0$. Now,
\[
p_4 = \frac{K(p(D)Q(x)^{-1})}{c_{4,4}}
= \frac{1}{384}K\left(\frac{\partial^4}{\partial t^3 \partial z} \left( \frac{1}{t^2 + x^2 + y^2 + z^2} \right) \right)
= \frac{1}{384}K\left(384t^3zQ(x)^{-5} - 144tzQ(x)^{-4} \right)
= K\left(t^3Q(x)^{-5} \right) - \frac{3}{8}K\left(tzQ(x)^{-4} \right)
= Q(x)^{-1}\left(\frac{t}{t^2 + x^2 + y^2 + z^2} \right)^3 \left(\frac{z}{t^2 + x^2 + y^2 + z^2} \right) Q\left(\frac{x}{Q(x)} \right)^{-5}
- \frac{3}{8}Q(x)^{-1}\left(\frac{tz}{(t^2 + x^2 + y^2 + z^2)^2} \right) Q\left(\frac{x}{Q(x)} \right)^{-4}
= t^3z - \frac{3}{8}tzQ(x)
\]
and hence since
\[ p = p_4 + Q(x)q_2, \]
we determine that $q_2 = \frac{3}{8}tz$. We do the same procedure to find $p_2$. Note that $q_2$ is already a harmonic polynomial, hence $p_2 = q_2$ and that $q_0 \equiv 0$ so that $p_0 \equiv 0$. In other words,
\[ p = \left(t^3z - \frac{3}{8}tzQ(x)\right) + Q(x)\left(\frac{3}{8}tz\right) + Q(x)^2(0) \]
is the desired decomposition.

We will give some more examples without giving the computation process.

**Example 10** If $p(x) = p(t, x, y, z) = \frac{1}{6}t^3$, then we can write $p$ as $p_3 + (t^2 + x^2 + y^2 + z^2)p_1$ where
\[ p_3 = \frac{1}{12}t^3 - \frac{1}{12}tx^2 - \frac{1}{12}ty^2 - \frac{1}{12}tz^2 \quad \text{and} \quad p_1 = \frac{1}{12}t. \]

Hence, on $S^3$,
\[ \frac{1}{6}t^3 = \frac{1}{12}t^3 - \frac{1}{12}tx^2 - \frac{1}{12}ty^2 - \frac{1}{12}tz^2 + \frac{1}{12}t. \]
Example 11 If \( p(x) = p(t, x, y, z) = \frac{1}{24}x^4 \), then we can write \( p \) as
\[
p = p_4 + (t^2 + x^2 + y^2 + z^2) p_2 + (t^2 + x^2 + y^2 + z^2)^2 p_0
\]
where
\[
p_4 = \frac{5}{384} t^4 + \frac{1}{384} x^4 + \frac{1}{384} y^4 + \frac{1}{384} z^4 - \frac{5}{192} t^2 x^2
\]
\[
- \frac{5}{192} t^2 y^2 - \frac{5}{192} t^2 z^2 + \frac{1}{192} x^2 y^2 + \frac{1}{192} x^2 z^2 + \frac{1}{192} y^2 z^2,
\]
\[
p_2 = -\frac{3}{64} t^2 + \frac{1}{64} x^2 + \frac{1}{64} y^2 + \frac{1}{64} z^2 \quad \text{and} \quad p_0 = -\frac{1}{64}.
\]

Thus, restricted to \( S^3 \),
\[
\frac{1}{24} x^4 = \frac{5}{384} t^4 + \frac{1}{384} x^4 + \frac{1}{384} y^4 + \frac{1}{384} z^4 - \frac{5}{192} t^2 x^2
\]
\[
- \frac{5}{192} t^2 y^2 - \frac{5}{192} t^2 z^2 + \frac{1}{192} x^2 y^2 + \frac{1}{192} x^2 z^2 + \frac{1}{192} y^2 z^2
\]
\[
- \frac{3}{64} t^2 + \frac{1}{64} x^2 + \frac{1}{64} y^2 + \frac{1}{64} z^2 - \frac{1}{64}.
\]

3.5 Bases for Harmonic Polynomials: Analytic Approach

In previous section we have seen that any polynomial of degree \( m \) can be written as a sum of homogeneous polynomials of degree \( 0 \leq k \leq m \), and restricted to the sphere, any homogeneous polynomial of degree \( k \) admits a nice decomposition into harmonic polynomials given in (3.7). Thus, one may view harmonic polynomials as building blocks for the space of polynomials on the sphere. It seems reasonable to consider bases for the space of harmonic polynomials on the sphere. We have followed a derivation that works for the field \( \mathbb{R} \), although clearly some aspects can be viewed more generally.

Recall from the Canonical Projection Theorem, that the canonical projection of \( P_k (\mathbb{A}^n) \) to \( H_k (\mathbb{A}^n) \) is given by
\[
p \mapsto K \left( p(D) Q(x)^{1 - \frac{1}{n}} \right)_{c_k,n}.
\]

For any homogeneous polynomial \( p \) of degree \( k \), we can write
\[
p = K \left( p(D) Q(x)^{1 - \frac{1}{n}} \right)_{c_k,n} + Q(x) q_{k-2},
\]
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for some \( q \in P_{k-2}(\mathbb{A}^n) \). If we restrict both sides to \( S^{n-1} \), we will get

\[
p|_S = \left[ \frac{K \left( p(D) Q(x)^{1-\frac{1}{2^n}} \right)}{c_{k,n}} + Q(x)q_{k-2} \right]_S
\]

\[
= \frac{K \left( p(D) Q(x)^{1-\frac{1}{2^n}} \right)}{c_{k,n}}|_S + q_{k-2}|_S
\]

\[
= \frac{p(D)Q(x)^{1-\frac{1}{2^n}}}{c_{k,n}} + q_{k-2}|_S. \tag{3.16}
\]

Recall from (3.9) that any two polynomials of different degrees will be orthogonal, with respect to the inner product (3.8). Combining with what we already developed so far in (3.16), we conclude that \( q_{k-2}|_S \) is orthogonal to \( H_k(S^{n-1}) \). We have now the following theorem by taking the orthogonal projection of both sides onto \( H_k(S^{n-1}) \).

**Theorem 15** The orthogonal projection of \( p|_S \) onto \( H_k(S^{n-1}) \) is

\[
p(D)Q(x)^{1-\frac{1}{2^n}} \quad \tag{3.17}
\]

**Corollary 5** Suppose \( n > 2 \) and \( p \in P_k(\mathbb{A}^n) \), then the set

\[
\left\{ K\left(D^\alpha Q(x)^{1-\frac{1}{2^n}} \right) : |\alpha| = k \right\}
\]

spans \( H_k(\mathbb{A}^n) \), and similarly, the set

\[
\left\{ D^\alpha Q(x)^{1-\frac{1}{2^n}} : |\alpha| = k \right\}
\]

spans \( H_k(S^{n-1}) \).

**Proof.** The proof is straightforward. The first statement follows by choosing the monomial \( p(x) = x^\alpha \) of degree \( k \). The second statement follows by restricting \( K\left(D^\alpha Q(x)^{1-\frac{1}{2^n}} \right) \) to \( S^{n-1} \) to get \( D^\alpha Q(x)^{1-\frac{1}{2^n}} \).

Given that these sets are spanning sets for \( H_k(\mathbb{A}^n) \) and \( H_k(S^{n-1}) \), respectively, one may want to find the basis of \( H_k(\mathbb{A}^n) \) and \( H_k(S^{n-1}) \) in terms of subsets of

\[
\left\{ K\left(D^\alpha Q(x)^{1-\frac{1}{2^n}} \right) : |\alpha| = k \right\}
\]

and

\[
\left\{ D^\alpha Q(x)^{1-\frac{1}{2^n}} : |\alpha| = k \right\},
\]

respectively.
Theorem 16 (Basis for $H_k (\mathbb{A}^n)$ and $H_k (S^{n-1})$) If $n > 2$, then the set
\[ \left\{ K \left( D^\alpha Q(x)^{1-\frac{1}{2}n} \right) : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\} \]
is a basis for $H_k (\mathbb{A}^n)$, and the set
\[ \left\{ D^\alpha Q(x)^{1-\frac{1}{2}n} : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\} \]
is a basis for $H_k (S^{n-1})$.

Proof. Let $B = \left\{ K \left( D^\alpha Q(x)^{1-\frac{1}{2}n} \right) : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\}$. We will show that $B$ spans $H_k (\mathbb{A}^n)$ by showing that $K \left( D^\alpha Q(x)^{1-\frac{1}{2}n} \right)$ is in the span of $B$ for every multi-index $\alpha$ of size $k$. If $\alpha_1 = 0$ or $\alpha_1 = 1$, then by the definition of $B$, we have that $K \left( D^\alpha Q(x)^{1-\frac{1}{2}n} \right) \in B$. We now proceed by induction on $\alpha_1$. Assume that $\alpha_1 > 1$ and that $K \left( D^\beta Q(x)^{1-\frac{1}{2}n} \right)$ is in the span of $B$ for all multi-indices $\beta$ of degree $k$ whose first components are less than $\alpha_1$. Since $Q(x)^{1-\frac{1}{2}n}$ is a harmonic function, then $\Delta Q(x)^{1-\frac{1}{2}n} = 0$, so we have that
\begin{align*}
K \left( D^\alpha Q(x)^{1-\frac{1}{2}n} \right) &= K \left( D_1^{\alpha_1-2} D_2^{\alpha_2} \cdots D_n^{\alpha_n} \left( D_1^2 Q(x)^{1-\frac{1}{2}n} \right) \right) \\
&= -K \left( D_1^{\alpha_1-2} D_2^{\alpha_2} \cdots D_n^{\alpha_n} \left( \sum_{i=2}^{n} D_i^2 Q(x)^{1-\frac{1}{2}n} \right) \right) \\
&= - \sum_{i=2}^{n} K \left( D_1^{\alpha_1-2} D_2^{\alpha_2} \cdots D_n^{\alpha_n} \left( D_i^2 Q(x)^{1-\frac{1}{2}n} \right) \right).
\end{align*}

By our induction hypothesis, each of the terms in the last line is in the span of $B$, therefore $B$ spans $H_k (\mathbb{A}^n)$.

Next it will be shown that
\[ |B| \leq \dim (H_k (\mathbb{A}^n)), \]
where $|B|$ denotes the cardinality of $B$. Since
\[ B = \left\{ K \left( D^\alpha Q(x)^{1-\frac{1}{2}n} \right) : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\}, \]
the number of elements in $B$ is at most the number of non-negative integer solutions of
\[ \alpha_1 + \alpha_2 + \cdots + \alpha_n = k, \]
where $\alpha_1 \leq 1$. If $\alpha_1 = 0$, then $\alpha_2 + \cdots + \alpha_n = k$ so by a basic combinatorial argument, the number of solution is
\[
\binom{k + (n - 1) - 1}{(n - 1) - 1} = \binom{n + k - 2}{n - 2}.
\]
If $\alpha_1 = 1$, then $\alpha_2 + \cdots + \alpha_n = k - 1$, so the number of the solutions is
\[
\binom{(k - 1) + (n - 1) - 1}{(n - 1) - 1} = \binom{n + k - 3}{n - 2}.
\]
Hence,
\[
|B| \leq \binom{k + n - 2}{n - 2} + \binom{k + n - 3}{n - 2} = \binom{n + k - 1}{n - 1} - \binom{n + k - 3}{n - 1} = \dim(H_k(\mathbb{R}^n)),
\]
where we have referred to Equation (3.10). Since $B$ is a spanning set for $H_k(\mathbb{R}^n)$, we have that $|B| \geq \dim H_k(\mathbb{R}^n)$, but at the same time $B \leq \dim H_k(\mathbb{R}^n)$, so we must have that
\[
|B| = \dim H_k(\mathbb{R}^n).
\]
This proves that $B$ is a basis for $H_k(\mathbb{R}^n)$. Now, since $B$ is a basis for $H_k(\mathbb{R}^n)$, by restricting every every element in $B$ in $S^{n-1}$, it follows that
\[
\left\{ D^\alpha Q(x)^{1-\frac{k}{n}} : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\}
\]
is a basis for $H_k(S^{n-1})$. \(\blacksquare\)

It is useful to see explicitly what the sets
\[
\left\{ K \left( D^\alpha Q(x)^{1-\frac{k}{n}} \right) : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\}
\]
and
\[
\left\{ D^\alpha Q(x)^{1-\frac{k}{n}} : |\alpha| = k \text{ and } \alpha_1 \leq 1 \right\}
\]
look like in the case $n = 4$. Below we give one example.

**Example 12** If $k = 2$, then $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$, so there are nine possibilities:
1. If $\alpha = (0, 2, 0, 0)$, then

\[
K \left( D^\alpha Q \left( x^{1-\frac{2}{n}} \right) \right) = K \left( D^2_2 Q \left( x^{-1} \right) \right)
\]
\[
= K \left( -2Q \left( x^{-2} \right) + 8x^2 Q \left( x^{-3} \right) \right)
\]
\[
= Q \left( x^{-1} \right) \left( -2Q \left( x^* \right)^{-2} + 8 \left( \frac{x}{Q(x)} \right)^2 Q \left( x^* \right)^{-3} \right)
\]
\[
= Q \left( x^{-1} \right) \left( -2Q \left( \frac{x}{Q(x)} \right)^{-2} + \frac{8x^2}{Q(x)^2} Q \left( \frac{x}{Q(x)} \right)^{-3} \right)
\]
\[
= -2Q \left( x \right) + 8x^2
\]
\[
= -2t^2 + 6x^2 - 2y^2 - 2z^2,
\]

2. If $\alpha = (0, 0, 2, 0)$, then similarly,

\[
K \left( D^\alpha Q \left( x^{1-\frac{2}{n}} \right) \right) = K \left( D^2_3 Q \left( x^{-1} \right) \right)
\]
\[
= -2t^2 - 2x^2 + 6y^2 - 2z^2,
\]

3. If $\alpha = (0, 0, 0, 2)$, then

\[
K \left( D^\alpha Q \left( x^{1-\frac{2}{n}} \right) \right) = K \left( D^2_4 Q \left( x^{-1} \right) \right)
\]
\[
= -2t^2 - 2x^2 - 2y^2 + 6z^2,
\]

4. If $\alpha = (0, 1, 1, 0)$ then

\[
K \left( D^\alpha Q \left( x^{1-\frac{2}{n}} \right) \right) = K \left( D_2D_3 Q \left( x^{-1} \right) \right)
\]
\[
= K \left( 8xyQ \left( x^{-3} \right) \right)
\]
\[
= Q \left( x^{-1} \right) \left( 8 \left( \frac{x}{Q(x)} \right) \left( \frac{y}{Q(x)} \right) Q \left( \frac{x}{Q(x)} \right)^{-3} \right)
\]
\[
= 8xy
\]

5. If $\alpha = (0, 1, 0, 1)$ then

\[
K \left( D^\alpha Q \left( x^{1-\frac{2}{n}} \right) \right) = K \left( D_2D_4 Q \left( x^{-1} \right) \right)
\]
\[
= 8xz
\]

6. If $\alpha = (0, 0, 1, 1)$ then

\[
K \left( D^\alpha Q \left( x^{1-\frac{2}{n}} \right) \right) = K \left( D_3D_4 Q \left( x^{-1} \right) \right)
\]
\[
= 8yz
\]
7. If $\alpha = (1, 1, 0, 0)$ then
\[
K \left( D^\alpha Q(x)^{1-\frac{1}{2n}} \right) = K \left( D_1 D_2 Q(x)^{-1} \right) \\
= 8tx,
\]

8. If $\alpha = (1, 0, 1, 0)$ then
\[
K \left( D^\alpha Q(x)^{1-\frac{1}{2n}} \right) = K \left( D_1 D_3 Q(x)^{-1} \right) \\
= 8ty,
\]

9. If $\alpha = (1, 0, 0, 1)$ then
\[
K \left( D^\alpha Q(x)^{1-\frac{1}{2n}} \right) = K \left( D_1 D_4 Q(x)^{-1} \right) \\
= 8tz.
\]

Hence a basis for $H_2(\mathbb{A}^4)$ is
\[
\left\{ -2t^2 + 6x^2 - 2y^2 - 2z^2, -2t^2 - 2x^2 + 6y^2 - 2z^2, \\
-2t^2 - 2x^2 - 2y^2 + 6z^2, 8xy, 8xz, 8yz, 8tx, 8ty, 8tz \right\}.
\]

Now since on $S^3$ we have $D_2^2 Q(x) = -2Q(x)^{-2} + 8x^2 Q(x)^{-3} = 8x^2 - 2$ and $8xyQ(x)^{-3} = 8xy$, then a basis for $H_2(S^3)$ is
\[
\left\{ 8x^2 - 2, 8y^2 - 2, 8z^2 - 2, 8xy, 8xz, 8yz, 8tx, 8ty, 8tz \right\}.
\]
Chapter 4

Theory of Harmonic Polynomials: Algebraic Approach

The standard theory of spherical functions that leads to the theory of harmonic polynomials developed in the previous chapter has given us a way to compare with the algebraic approach we will shortly see in this chapter. We will first discuss harmonic polynomials on $S^2$ by using the idea of a factorial basis which then can be arranged geometrically using triangular patterns in the plane. This approach was first introduced by Wildberger [31] in his preprint to study spherical harmonic polynomials on a low-dimensional sphere. The core of this thesis is to extend the theory to the case of $S^3$ to explain the associated harmonic polynomials purely in an algebraic manner. Finally, we will talk about the rotationally invariant harmonic polynomials on $S^3$ which may be interpreted as characters on the Lie group $S^3 \simeq SU(2)$. Most of the results here are more general, and work also over the rational numbers, or more general field.

To start with, we remind the reader about the concept of factorial and double factorial. They will be used quite often throughout this chapter and in Appendix B.

**Definition 5** For any non-negative integer $k$, the factorial of $k$, denoted by $k!$, is given by

$$k! = k \times (k - 1) \times \cdots \times 1$$

if $k$ is positive, and we define $0! = 1$. The double factorial of $k$, denoted by $k!!$,
is given by
\[
\begin{cases}
  k!! = k \times (k-2) \times (k-4) \times \cdots \times 1 & \text{if } k \text{ is odd,} \\
  k \times (k-2) \times (k-4) \times \cdots \times 2 & \text{if } k \text{ is even}
\end{cases}
\]
for \( k > 0 \), and we also define \( 0!! \) to be 1.

The factorial of \( k \) can be thought of as the function from the set of non-negative integers \( \mathbb{Z}_{\geq 0} \) to itself. Over the field \( \mathbb{R} \), there is an analytic continuation of the factorial function so that at \( \xi = k \), it gives the value of \( k! \), but we can define the value at non-integer value of \( x \). See Artin [5, Chapter 2] or Rudin [26, Chapter 8] for further explanations.

**Definition 6** If \( F = \mathbb{R} \), there exists a continuous function \( \Gamma \) that satisfies \( \Gamma(0) = 1 \) and \( \Gamma(x+1) = x\Gamma(x) \) for \( x \in \mathbb{R} \). Such a function is called the gamma function.

We will show that the gamma function \( \Gamma \) is related to the double factorial function.

**Proposition 7** If \( k \) is a positive integer, then
\[
\Gamma\left(k + \frac{1}{2}\right) = \frac{(2k - 1)!!}{2^k} \Gamma\left(\frac{1}{2}\right).
\]

**Proof.** The proof is simple and uses the basic property of gamma function. Note that since \( \Gamma(x+1) = x\Gamma(x) \) for any \( x \in \mathbb{R} \), we must have that, for \( k \geq 1 \),
\[
\Gamma\left(k + \frac{1}{2}\right) = \left(k - \frac{1}{2}\right) \Gamma\left(k - \frac{1}{2}\right).
\]
Recursively, we also have that
\[
\Gamma\left(k - \frac{1}{2}\right) = \left(k - \frac{3}{2}\right) \Gamma\left(k - \frac{3}{2}\right),
\]
so that
\[
\Gamma\left(k + \frac{1}{2}\right) = \left(k - \frac{1}{2}\right) \left(k - \frac{3}{2}\right) \Gamma\left(k - \frac{3}{2}\right).
\]
If we follow this process, after a finite number of times we will have that
\[
\Gamma\left(k + \frac{1}{2}\right) = \left(k - \frac{1}{2}\right) \left(k - \frac{3}{2}\right) \cdots \Gamma\left(\frac{1}{2}\right) = \frac{(2k - 1)!!}{2^k} \Gamma\left(\frac{1}{2}\right),
\]
for \( k > 0 \).
4. Theory of Harmonic Polynomials: Algebraic Approach

4.1 Harmonic Polynomials on $S^2$

We are now going to use a slightly modified notation for monomials which will turn out to be very useful. This was introduced in Wildberger’s preprint [31].

**Definition 7** For any integer $k$, and any variable $x$, define the notation

$$x^k = \frac{1}{k!}x^k.$$  

We define the term $\frac{1}{k!}$ to be zero if $k$ is negative.

Since we are going to investigate polynomials $p(x)$ where $x \in S^2 \subset \mathbb{A}^3$, it is convenient to let $x = (x, y, z)$ instead of $(x_1, x_2, x_3)$. Denote by $x^a y^b z^c$ the monomial

$$\frac{x^a y^b z^c}{a!b!c!},$$

and observe that whenever either $a$, $b$, or $c$ is negative, then the above monomial will be automatically zero.

Note that

$$\{ x^a y^b z^c : a + b + c = k \text{ and } a, b, c \geq 0 \}$$

gives us a basis for $P_k (\mathbb{A}^3)$. We will call this basis the **factorial basis** of $P_k (\mathbb{A}^3)$. We can arrange these basis monomials in a triangle which will be denoted as the **level-$k$ triangle**, as shown in the following examples for $k = 1$, $2$, and $4$. Here we set the convention that the distance between each vertex on the side of the triangle is 1. Figure (4.1) shows the level-1 triangle in factorial basis. The next one shows the level-2 triangle and level-4 triangle, as shown in Figure (4.2) and Figure (4.3).

Any homogeneous polynomial of degree $k$ can be identified by a level-$k$ triangle of coefficients with respect to the factorial basis of $P_k (\mathbb{A}^n)$. For example look at the triangle in Figure (4.4) below.

The triangle in Figure (4.4) corresponds to the polynomial

$$p(x, y, z) = x^2 - 3xy + 2y^2 - 2xz + 4yz + 4z^2$$

$$= \frac{1}{2}(x^2 - 3xy + y^2 - 2xz + 4yz + 2z^2).$$

The action of the Laplacian $\Delta$ is particularly pleasant in the factorial basis.
Lemma 4 For any integer $a, b, c$,

$$\Delta x^a y^b z^c = x^{a-2} y^b z^c + x^a y^{b-2} z^c + x^a y^b z^{c-2}. \quad (4.1)$$

Proof. The proof is computational. The equality in (4.1) is obviously true when either one of $a, b, c$ is less than 2. Thus, we may safely assume that all of them are at least 2. We have

$$\Delta x^a y^b z^c = \frac{1}{a! b! c!} \Delta x^a y^b z^c$$

$$= \frac{1}{a! b! c!} (a (a - 1) x^{a-2} y^b z^c + b (b - 1) x^a y^{b-2} z^c + c (c - 1) x^a y^b z^{c-2})$$

$$= \frac{1}{(a - 2)! b! c!} x^{a-2} y^b z^c + \frac{1}{a! (b - 2)! c!} x^a y^{b-2} z^c + \frac{1}{a! b! (c - 2)!} x^a y^b z^{c-2}$$

$$= x^{a-2} y^b z^c + x^a y^{b-2} z^c + x^a y^b z^{c-2},$$

which completes the proof. ■
Now we give a geometrical interpretation of the Lemma 4, which was described in Wildberger [31]. First imagine stacking the level-\(k\) triangles of factorial basis elements successively below each other to form a pyramidal packing with \(k = 0\) at the top, \(k = 1\) beneath that, \(k = 2\) beneath that, and so on. If each element is considered to be contained in a sphere, we get a regular spherical packing with each interior sphere touching 6 adjacent spheres on its own level, 3 spheres in the level beneath it and 3 spheres in the level above it.

Lemma 4 tells that if we have a homogeneous polynomial \(p\) of degree \(k\), represented in the factorial basis as

\[
p(x, y, z) = \sum_{a+b+c=k} \lambda_{a,b,c} x^a y^b z^c,
\]

then for each monomial \(x^a y^b z^c\), the Laplacian applied to that monomial sends the coefficient \(\lambda_{a,b,c}\) in the level-\(k\) triangle to the three positions corresponding to the monomials \(x^{a-2}y^b z^c\), \(x^a y^{b-2} z^c\), and \(x^a y^b z^{c-2}\) in the level-(\(k-2\)) triangle 2 levels above it. This leads to the conclusion that \(p\) is harmonic precisely when in the level-\(k\) triangle representation of \(p\), the sum of all coefficients in every subtriangle of side 2 is zero.

For example, in the case of the polynomial presented in Figure (4.4) above, the only subtriangle of side 2 is the triangle with vertices \(x^2, y^2,\) and \(z^2\), with the corresponding coefficients 1, 2, and 4. Since \(1 + 2 + 4 \neq 0\), then \(p\) is not harmonic. Let us give one more example.
4. Theory of Harmonic Polynomials: Algebraic Approach

In the above level-4 triangle, there are 6 subtriangles with side 2 and in every subtriangle, the sum of the coefficients corresponding to each vertex must be zero in order to be harmonic. In this particular example, if

$$p(x, y, z) = a_1 z^4 + a_2 xz^3 + a_3 yz^3 + a_4 x^2 z^2 + a_5 xyz^2 + a_6 y^2 z^2 + a_7 x^3 z + a_8 x^2 yz + a_9 xy^2 z + a_{10} y^3 z + a_{11} x^4 + a_{12} x^3 y + a_{13} x^2 y^2 + a_{14} xy^3 + a_{15} y^4,$$

then $p$ will be harmonic precisely when

$$
\begin{align*}
& a_1 + a_4 + a_6 = 0, \\
& a_2 + a_7 + a_9 = 0, \\
& a_3 + a_8 + a_{10} = 0, \\
& a_4 + a_{11} + a_{13} = 0, \\
& a_5 + a_{12} + a_{14} = 0, \\
& a_6 + a_{13} + a_{15} = 0.
\end{align*}
$$

In this particular example, we have 15 variables and 6 equations. That gives 9 degrees of freedom, so the dimension of $H_4(A^3)$ must be 9. In general, in the level-$k$ triangle, the number of variables is just $1 + 2 + \cdots + (k + 1) =$
4. Theory of Harmonic Polynomials: Algebraic Approach

Figure 4.5: Level 4 triangle with coefficients in factorial basis

\[ \frac{1}{2} (k + 1) (k + 2) \text{ and there are } \frac{1}{2} k (k - 1) \text{ subtriangles with side } 2, \text{ so} \]

\[ \dim H_k (\mathbb{A}^3) = \frac{1}{2} (k + 1) (k + 2) - \frac{1}{2} k (k - 1) \]

\[ = 2k + 1. \quad (4.2) \]

This can be justified as well by putting \( n = 3 \) to Equation (3.10), so this gives the geometric meaning to the expression \( \dim (H_k (\mathbb{A}^n)) \).

4.1.1 Complex Factorial Basis

Now, introduce two variables

\[ u = \frac{x + iy}{2} \text{ and } v = \frac{x - iy}{2} = \overline{u}. \]

With this substitution, since we can express \( x \) and \( y \) in terms of \( u \) and \( v \), it is clear that any homogeneous polynomial \( p \) in \( x, y, \) and \( z \) can be not only expressed in terms of \( u, v, \) and \( z \), but they also have the same degree as \( p \). It means that the set

\[ \{ u^a v^b z^c : a + b + c = k \} \]
Figure 4.6: Level 2 triangle in complex factorial basis

serves as a basis for $P_k(\mathbb{A}^3)$. We call it the complex factorial basis of $P_k(\mathbb{A}^3)$. We can still arrange the elements of this set into a level-$k$ triangle as before, as shown in Figure (4.6). The only difference is that $x$ and $y$ are replaced by $u$ and $v$, respectively.

As for the action of the Laplacian of any element in the complex factorial basis, it is even simpler.

**Lemma 5** For any integer $a$, $b$, and $c$, we have

$$
\Delta u^a v^b z^c = u^{a-1} v^{b-1} z^c + u^a v^b z^{c-2}.
$$

**Proof.** Note that

$$
\frac{\partial^2}{\partial x^2} (u^a v^b z^c) = \frac{z^c}{a!b!c!} \frac{\partial^2}{\partial x^2} (u^a v^b) = \frac{z^c}{a!b!c!} \left( \frac{a(a-1)}{4} u^{a-2} v^b + \frac{ab}{2} u^{a-1} v^{b-1} + \frac{b(b-1)}{4} u^a v^{b-2} \right)
$$

and by the same principle,

$$
\frac{\partial^2}{\partial y^2} (u^a v^b z^c) = \frac{z^c}{a!b!c!} \left( -\frac{a(a-1)}{4} u^{a-2} v^b + \frac{ab}{2} u^{a-1} v^{b-1} - \frac{b(b-1)}{4} u^a v^{b-2} \right)
$$

so that

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (u^a v^b z^c) = \frac{z^c}{a!b!c!} (abu^{a-1} v^{b-1}) = \frac{u^{a-1} v^{b-1} z^c}{(a-1)! (b-1)! c!} = u^{a-1} v^{b-1} z^c.
$$
Now since
\[ \frac{\partial^2}{\partial z^2} \left( u^a v^b z^c \right) = u^a v^b \frac{\partial^2}{\partial z^2} \left( \frac{1}{c!} z^c \right) = u^a v^b \left( \frac{z^{c-2}}{(c-2)!} \right) = u^a v^b \frac{z^{c-2}}{c-2} , \]
it follows that
\[ \Delta u^a v^b z^c = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u^a v^b z^c \]
\[ = u^{a-1} v^{b-1} z^c + u^a v^b z^{c-2} , \]
as desired. \[ \blacksquare \]

**Remark 6** Note that the proof above implicitly implies that
\[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial u \partial v} \]
since for every monomial \( u^a v^b z^c \), we have that
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (u^a v^b z^c) = u^{a-1} v^{b-1} z^c = \frac{\partial^2}{\partial u \partial v} (u^a v^b z^c) . \]

Lemma 5 tells us that if we represent a homogeneous polynomial \( p \in P_k (\mathbb{A}^3) \) by its triangle of coefficients in the complex factorial basis, then \( p \) is harmonic if and only if in every vertical column the entries have constant absolute value and are alternating in sign as we move up (or down). The basis elements of \( H_k (\mathbb{A}^3) \) are then represented by vertical columns in the level-\( k \) triangle, consisting of alternating plus and minus ones. A vertical column in called **non-degenerate** if it consists of more than one entries. A vertical column that is not non-degenerate is called **degenerate**. Looking back at Figure (4.6), there are five vertical columns, the ones that start with \( u^2, u z, z^2, v z, \) and \( v^2 \). All of the columns other than that which starts with \( z^2 \) consist of just one term which is itself. The column that starts with \( z^2 \) represents the polynomial \( z^2 - uv \). Consider Figure (4.7). The only non-degenerate vertical column is marked with an arrow and that corresponds to \( z^2 - uv \).

Hence, a basis for \( H_2 (\mathbb{A}^3) \) is
\[ \{ u^2, u z, z^2 - uv, v z, v^2 \} . \]
4. Theory of Harmonic Polynomials: Algebraic Approach

By the same reasoning, there are seven vertical columns in the level-3 triangle in the complex factorial basis. All of the columns other than which start with \( u^2, z^3, \) and \( vz^2 \) consist of just one term which is itself. The column that starts with \( u^2, z^3, \) and \( vz^2 \) represents the basis element \( u^2 - u^2 v, z^3 - uvz, \) and \( vz^2 - uv^2, \) respectively. Hence a basis for \( H_3(\mathbb{A}^3) \) is

\[
\left\{u^3, \ u^2 z, \ u^2 - u^2 v, \ z^3 - uvz, \ vz^2 - uv^2, \ v^2 z, \ v^3\right\}.
\]

We provide one more example. There are nine vertical columns in the level-4 triangle in the complex factorial basis. We mark the columns consisting of two and three entries by arrows, so the columns with two entries correspond to polynomials \( u^2 z^2 - u^3 v, \ u z^3 - u^2 v z, \ vz^3 - uw^2 z, \) and \( v^2 z^2 - w^2 u. \) The column consisting of three entries corresponds to \( z^4 - uvz^2 + u^2 v^2. \) The other four vertical columns are degenerate. See Figure (4.8).

Hence a basis for \( H_4(\mathbb{A}^3) \) is

\[
\left\{u^4, \ u^3 z, \ u^2 z^2 - u^3 v, \ u z^3 - u^2 v z, \ vz^3 - uvz^2 + u^2 v^2, \ vz^3 - uw^2 z, \ v^2 z^2 - uv^3, \ v^3 z, \ v^4\right\}.
\]

Let \( (u^a v^b z^c) \) be the vertical column in the level-\((a + b + c)\) triangle in complex factorial basis that has \( u^a v^b z^c \) as one of its entries. Thus, \( (u^a v^b z^c) \) is the same column as \( (u^{a+1} v^{b+1} z^{c-2}) \). Denote by \([u^a v^b z^c]\) the polynomial that is represented by \((u^a v^b z^c)\). We have the following Theorem.
Figure 4.8: Columns in level 4 triangle in complex factorial basis

**Theorem 17** A basis for $H_k(A^3)$ is

\[ \left\{ [u^k], [u^{k-1}z], [u^{k-2}z^2], \ldots, [uz^{k-1}], \ldots, [v^{k-2}z^2], [v^{k-1}z], [v^k] \right\}. \]

### 4.1.2 Zonal Harmonic Polynomials on $S^2$

In this subsection, we are going to study one special case of harmonic polynomials on $S^2$, called the zonal harmonic polynomials. To do that, we need a few concepts in hand first. Let $A^3$ be the usual three-dimensional affine space over a field $F$ and a particular point $x \in A^3$ be $(x, y, z)$. In the classical theory when $F = \mathbb{R}$, the parametrization of the sphere $S^2$ in the Euclidean space $\mathbb{R}^3$ distinguishes the $z$ direction. Let

\[ h(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
be the subgroup of the rotation matrices in three-dimensional space \( \text{SO}(3) \). We say a polynomial \( u \) on \( P(\mathbb{R}^3) \) transforms by \( e^{i\theta} \) if
\[
 u \left( \mathbf{x} \left( \theta \right)^T \right) = e^{i\theta} u \left( \mathbf{x} \right)
\]
for all \( \theta \in [0, 2\pi) \) and all relevant \( \mathbf{x} = (x, y, z) \). Here \( l \in \mathbb{Z} \). The case \( l = 0 \) means that \( u \) is unchanged by the rotation \( h(\theta) \).

**Definition 8** A polynomial \( u \in P(\mathbb{R}^3) \) is called **zonal** precisely when \( u \) is unchanged by the rotation \( h(\theta) \); in other words,
\[
 u \left( \mathbf{x} \left( \theta \right)^T \right) = u \left( \mathbf{x} \right)
\]
for every \( \mathbf{x} \in \mathbb{R}^3 \).

Now, define
\[
P_k^l \left( \mathbb{R}^3 \right) = \left\{ p \in P_k \left( \mathbb{R}^3 \right) : p \left( \mathbf{x} \left( \theta \right)^T \right) = e^{i\theta} p \left( \mathbf{x} \right) \text{ for all } \theta \in [0, 2\pi) \right\}.
\]
Since \( P_k \left( \mathbb{R}^3 \right) \) is finite-dimensional, we must have
\[
P_k^l \left( \mathbb{R}^3 \right) = \bigoplus_{l \in \mathbb{Z}} P_k^l \left( \mathbb{R}^3 \right). \tag{4.4}
\]
However, in the following Lemma we will see that the infinite direct sum in (4.4) only has finitely many nonzero term.

**Lemma 6** For any \( k \in \mathbb{N} \),
\[
P_k^l \left( \mathbb{R}^3 \right) = \bigoplus_{l=-k}^{k} P_k^l \left( \mathbb{R}^3 \right).
\]

**Proof.** Suppose \( p \) is of degree \( k < |l| \). Then if we see the polynomial \( p \left( \mathbf{x} \left( \theta \right)^T \right) \) as a polynomial of \( \theta \), then \( p \) is a trigonometric polynomial of degree at most \( k \) in terms of \( \cos \theta \) and \( \sin \theta \). Such a polynomial will be orthogonal to \( e^{i\theta} \) under the usual inner product
\[
\langle p, e^{i\theta} \rangle = \frac{1}{2\pi} \int_0^{2\pi} p(\theta) e^{i\theta} d\theta,
\]
so \( P_k^l = \{0\} \) if \( k < |l| \). \( \blacksquare \)
Since the zonal polynomials are unchanged by the rotation along the $z$-axis, then combined with the theory of harmonic polynomials developed in the previous subsection, the unique zonal harmonic homogeneous polynomial of degree $k$ must correspond to $\left[ z^k \right]$. Call this zonal harmonic $p_k$, then we have that

$$p_k = z^k - uvz^{k-2} + u^2v^2z^{k-4} - \cdots$$  \hspace{1cm} (4.5)

which ends in $(-1)^m u^m v^m z$ if $k$ is odd, or $u^m z^m$ if $k$ is even. Here $m = \left\lfloor \frac{k}{2} \right\rfloor$. We can rewrite (4.5) as

$$p_k = \sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^l uv^l z^{k-2l}.$$  \hspace{1cm} (4.6)

If we expand the relation above and resubstitute $x$ and $y$ back, we have that

$$p_k = \sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^l \frac{1}{l!!(k-2l)!} (uv)^l z^{k-2l}.$$  \hspace{1cm} (4.7)

Here are the first few values of $p_k$:

$$p_0 = 1,$$

$$p_1 = z,$$

$$p_2 = z^2 - uv = \frac{1}{2}z^2 - \frac{1}{4}(x^2 + y^2),$$

$$p_3 = z^3 - uvz = \frac{1}{6}z^3 - \frac{1}{4}(x^2 + y^2) z,$$

$$p_4 = z^4 - uvz^2 + u^2v^2 = \frac{1}{24}z^4 - \frac{1}{8}(x^2 + y^2) z^2 + \frac{1}{64}(x^2 + y^2)^2.$$

Now, for each $k$, we can express $p_k$ as a polynomial in a single variable $z$ because on $S^2$ we have $x^2 + y^2 = 1 - z^2$. Another way of describing the zonal harmonic $p_k$ is by rewriting the Equation (4.7) as

$$p_k = \sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \left( -\frac{1}{4} \right)^l \frac{1}{l!!(k-2l)!} (1 - z^2)^l z^{k-2l}.$$  \hspace{1cm} (4.8)

Hence, the first few values of $p_k$ in terms of $z$ only are
The polynomials $p_k$ above have a neat description in terms of Legendre polynomials, one of many special polynomials. Here we present our own proof to the following important Theorem.

**Theorem 18** For each $k \geq 0$, the polynomial $p_k$ above satisfies

$$p_k(z) = \frac{1}{k!} P_k(z),$$

where $P_k(z)$ is the $k$-th Legendre polynomial.

The proof of the Theorem makes use of the following Lemma.

**Lemma 7** For any non-negative integer $k$ and non-negative integer $m$ such that $m \leq \left\lfloor \frac{k}{2} \right\rfloor$, we have that

$$\sum_{l=m}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{k!}{4^l l! (k-2l)! (l-m)!} = \frac{(2k-2m)!}{2^k (k-m)! (k-2m)!}.$$ 

**Proof.** The equation is definitely true for $k = 0$. Hence, assume that $k \geq 1$. By shifting the index $l$ in the summation in the left hand side, we get that

$$\sum_{l=m}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{k!}{4^l l! (k-2l)! (l-m)!} = \sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor - m} \frac{k!}{4^l + m (l+m)! (k-2l-2m)!!}.$$ 

Now we want to follow Zeilberger’s approach in finding a neat formula to the above sum in terms of hypergeometric functions. The algorithm is given in [24, Chapter 3]. Since the first term in the above sum is not 1, we write

$$\sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor - m} \frac{k!}{4^l + m (l+m)! (k-2l-2m)!!} = \frac{k!}{4^m m! (k-2m)!} \sum_{l=0}^{m} \frac{m! (k-2m)!}{4^l (l+m)! (k-2l-2m)!!}.$$
so that the first entry in the last sum is 1. Now we can express the sum
\[
\sum_{l=0}^{\left\lfloor \frac{m}{2} \right\rfloor - m} \frac{m! (k - 2m)!}{4^l (l + m)! (k - 2l - 2m)!!}
\]
(4.9)
in terms of a hypergeometric function. Define
\[
a_l = \frac{m! (k - 2m)!}{4^l (l + m)! (k - 2l - 2m)!!},
\]
so that
\[
\frac{a_{l+1}}{a_l} = \frac{4^l (l + m)! (k - 2l - 2m)!!}{4^{l+1} (l + 1 + m)! (k - 2l - 2m - 2)! (l + 1)!}
= \frac{(k - 2l - 2m) (k - 2l - 2m - 1)}{4 (l + m + 1) (l + 1)}
= \frac{(l - k - 2m) (l - k - 2m - 1)}{(l + m + 1) (l + 1)}.
\]
Hence the sum (4.9) corresponds to the hypergeometric function
\[
\frac{k!}{4^m m! (k - 2m)!} \, _2F_1 \left( -\frac{1}{2} (k - 2m) , -\frac{1}{2} (k - 2m - 1) ; m + 1 ; 1 \right).
\]
(4.10)
Recall Gauss’ \(_2F_1\) identity given below for rational numbers \(a\), \(b\), and \(c\). See, for instance Gradshteyn [16, Chapter 9] or Andrews [4, Chapter 2].
\[
_2F_1 (a, b; c; 1) = \frac{\Gamma (c - a - b) \Gamma (c)}{\Gamma (c - a) \Gamma (c - b)}
\]
(4.11)
as long as \(c - a - b > 0\). In this case,
\[
a = -\frac{1}{2} (k - 2m) , \quad b = -\frac{1}{2} (k - 2m - 1) , \quad \text{and} \quad c = m + 1.
\]
Since \(c - a - b = k - m + \frac{1}{2} > 0\), we must have that
\[
_2F_1 \left( -\frac{1}{2} (k - 2m) , -\frac{1}{2} (k - 2m - 1) ; m + 1 ; 1 \right) = \frac{\Gamma (k - m + \frac{1}{2}) \Gamma (m + 1)}{\Gamma (\frac{1}{2} k + 1) \Gamma (\frac{1}{2} k + \frac{1}{2})},
\]
so Equation (4.10) becomes
\[
\frac{k!}{4^m m! (k - 2m)!} \times \frac{\Gamma (k - m + \frac{1}{2}) \Gamma (m + 1)}{\Gamma (\frac{1}{2} k + 1) \Gamma (\frac{1}{2} k + \frac{1}{2})}
= \frac{k!}{4^m m! (k - 2m)!} \times \frac{\Gamma (k - m + \frac{1}{2}) m!}{\Gamma (\frac{1}{2} k + 1) \Gamma (\frac{1}{2} k + \frac{1}{2})}
= \frac{k!}{4^m (k - 2m)!} \times \frac{\Gamma (k - m + \frac{1}{2})}{\Gamma (\frac{1}{2} k + 1) \Gamma (\frac{1}{2} k + \frac{1}{2})}.
\]
(4.12)
Proposition 7 states that if \( k \) is a positive integer, we have that

\[
\Gamma \left( k + \frac{1}{2} \right) = \frac{(2k - 1)!!}{2^k} \Gamma \left( \frac{1}{2} \right).
\]

We divide into 2 cases. If \( k \) is even, write \( k = 2p \) for some \( p \in \mathbb{Z} \). Hence we have that Equation (4.12) reduces to

\[
\frac{k!}{4^m (k-2m)!} \times \frac{(2k - 2m - 1)!! \Gamma \left( \frac{1}{2} \right)}{2^{k-m}} \times \frac{1}{\Gamma (p + 1) \Gamma (p + \frac{1}{2})} = \frac{k!}{4^m (k-2m)!} \times \frac{(2k - 2m - 1)!! \Gamma \left( \frac{1}{2} \right)}{2^{k-m}} \times \frac{1}{p! (2p - 1)!! \Gamma \left( \frac{1}{2} \right)}
\]

\[
= \frac{1}{(k-2m)!} \times \frac{2^{2m+k-m-p} p! (k-1)!}{(2k - 2m - 1)!} \times (2k - 2m - 1)!
\]

\[
= \frac{1}{2^m (k-2m)!} \times \frac{k \times (k-2) \cdots 2}{2p!} \times (2k - 2m - 1)!
\]

\[
= \frac{1}{2^m (k-2m)!} \times \frac{2p \times (2p - 2) \cdots 2}{2p!} \times \frac{(2k - 2m)!}{(2k - 2m) \times (2k - 2m - 2) \times \cdots \times 2}
\]

\[
= \frac{1}{2^m (k-2m)!} \times \frac{p \times (p-1) \cdots 1}{p!} \times \frac{(2k - 2m)!}{2^{k-m} (k-m) \times (k-m-1) \times \cdots \times 1}
\]

\[
= \frac{1}{2^m (k-2m)!} \times \frac{(2k - 2m)!}{(2k - 2m)! \times (k-m)!}
\]

as desired.

If \( k \) is odd, write \( k = 2p + 1 \) for some \( p \in \mathbb{Z} \). Hence the Equation (4.12) simplifies to

\[
\frac{k!}{4^m (k-2m)!} \times \frac{(2k - 2m - 1)!! \Gamma \left( \frac{1}{2} \right)}{2^{k-m}} \times \frac{1}{\Gamma (p + \frac{3}{2}) \Gamma (p + 1)} = \frac{k!}{4^m (k-2m)!} \times \frac{(2k - 2m - 1)!! \Gamma \left( \frac{1}{2} \right)}{2^{k-m}} \times \frac{1}{(2p + 1)!! \Gamma \left( \frac{1}{2} \right)}
\]

\[
= \frac{1}{2^{2m} (k-2m)!} \times \frac{(2p + 1)!}{(2p + 1)!!} \times (2k - 2m - 1)!
\]

\[
= \frac{1}{2^m (k-2m)!} \times \frac{(2p) \times (2p - 2) \cdots 2}{2p!} \times (2k - 2m - 1)!
\]
which, as above, reduces to
\[
\frac{(2k - 2m)!}{2^k (k - 2m)! (k - m)!}.
\]
Since the statement is true for the case \(k\) even and \(k\) odd, we conclude that
\[
\sum_{l=m}^{2l} \frac{k!}{4^l (k - 2l)! (l - m)!} = \frac{(2k - 2m)!}{2^k (k - m)! (k - 2m)!},
\]
as desired.

**Proof of Theorem 18.** Let \(A(z) = k! p_k(z)\). Based on (4.8), we have that
\[
A(z) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \left( \frac{-1}{4} \right)^l \frac{k!}{l!(k - 2l)!} (1 - z^2)^l z^{k-2l}
\]

\[
= \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{4^l (l!)^2 (k - 2l)!} (z^2 - 1)^l z^{k-2l}
\]

\[
= \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{4^l (l!)^2 (k - 2l)!} \left( \sum_{m=0}^{l} \binom{l}{m} (-1)^m z^{2l-2m} \right) z^{k-2l}
\]

\[
= \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=0}^{l} \frac{(-1)^m k!}{4^l (l!)^2 (k - 2l)!} \binom{l}{m} z^{k-2m}
\]

By changing the order of summation, we will have
\[
A(z) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=m}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m k!}{4^l (k - 2l)! (l - m)!} z^{k-2m}
\]

\[
= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m z^{k-2m}}{m!} \left( \sum_{l=m}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{4^l (k - 2l)! (l - m)!} \right).
\] (4.13)

Incorporating Lemma 7 above, we see that the Equation (4.13) above reduces to
\[
A(z) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m z^{k-2m}}{m!} \frac{(2k - 2m)!}{2^k (k - m)! (k - 2m)!}
\]

\[
= \frac{1}{2^k} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \frac{(2k - 2m)!}{m! (k - m)! (k - 2m)!} z^{k-2m},
\]
which is one of the representation of Legendre polynomials. See Silverman [28, Chapter 4] for further references.

### 4.1.3 Generating Function

Now that we have explicitly found our zonal harmonic polynomials \( p_k \) on \( S^2 \), it is interesting to consider the generating function of \( p_k \); that is, we would like to find an closed, explicit form of

\[
F(w) = 
\sum_{k=0}^{\infty} p_k w^k.
\]

Based on the formula of \( p_k \) given in (4.5), we get

\[
F(w) = 1 + zw + \left( \frac{z^2 - uw}{w} \right) w^2 + \left( \frac{z^3 - uwz}{w^2} \right) w^3 + \left( \frac{z^4 - uwz^2 + w^2u^2}{w^3} \right) w^4 + \cdots
\]

\[
= (1 + zw + \frac{z^2w^2 + z^3w^3 + \cdots}{w^2}) \left( 1 - \frac{uw^2 + w^2u^2 - \cdots}{w^2} \right)
\]

\[
= \sum_{k=0}^{\infty} \frac{z^k}{w^k} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (uv)^k}{k!} w^{2k} \right)
\]

\[
= e^{zw} \sum_{k=0}^{\infty} \frac{(-1)^k (uv)^k}{k!k!} w^{2k}.
\]

The second sum is similar in form with Bessel function of zero order. See Bowman [8, Chapter 1] for explicit formula or Andrews [4, p200] for formula in terms of hypergeometric functions. It is given by

\[
J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left( \frac{x}{2} \right)^{2k}
\]

where \( x = 2w\sqrt{uv} \). Hence,

\[
F(w) = e^{zw} J_0 \left( 2\sqrt{uvw} \right)
\]

\[
= e^{zw} J_0 \left( \sqrt{1 - z^2w} \right).
\]

By Theorem 18, we know that \( p_k(z) \) can be written in terms of Legendre polynomials. Thus, we have the following corollary.
Corollary 6  The exponential generating function of Legendre polynomial involves Bessel function of zero order. More precisely,
\[ \sum_{k=0}^{\infty} \frac{P_k(z)}{k!} w^k = e^w J_0 \left( \sqrt{1 - z^2 w} \right), \]
where \( P_k(z) \) is the \( k \)th Legendre polynomial, \( k = 0, 1, 2, \ldots \).

This adds to the collection of generating functions of Legendre polynomials. The standard generating function of Legendre polynomials is given by
\[ \sum_{k=0}^{\infty} P_k(z) w^k = \frac{1}{\sqrt{1 - 2zw + w^2}}. \]

4.2 Harmonic Polynomials on \( S^3 \)

Now we move to the case when \( n = 4 \); that is, we consider harmonic polynomials \( p(x) \) where \( x \in S^3 \subset A^4 \). Again, for convenience, we write \( x = (t, x, y, z) \) instead of \((x_1, x_2, x_3, x_4)\). We denote by \( t^a x^b y^c z^d \) the factorial monomial
\[ \frac{t^a x^b y^c z^d}{a! b! c! d!}, \]
for any integer \( a, b, c, d \). Again, if either one of \( a, b, c, \) or \( d \) is negative, the above expression will be automatically zero.

The following lemma is analogous to the Lemma in our previous section given in (4.1), and the proof is similar.

Lemma 8  For any integer \( a, b, c, d, \)
\[ \Delta \left( t^a x^b y^c z^d \right) = \frac{t^{a-2} x^b y^c z^d}{a! b! c! d!} + \frac{t^a x^{b-2} y^c z^d}{a! b! c! d!} + \frac{t^a x^b y^{c-2} z^d}{a! b! c! d!} + \frac{t^a x^b y^c z^{d-2}}{a! b! c! d!}. \quad (4.15) \]

Apart from the standard basis
\[ \{t^a x^b y^c z^d : a + b + c + d = k \quad \text{and} \quad a, b, c, d \in \mathbb{Z}\} \]
of \( P_k(A^4) \),
\[ \{t^a x^b y^c z^k : a + b + c + d = k \quad \text{and} \quad a, b, c, d \in \mathbb{Z}\} \]
is also a basis of $P_k(\mathbb{A}^4)$, which we will call \textbf{factorial basis} of $P_k$. We can arrange all these factorial basis monomials in a tetrahedron which will be referred as the \textbf{level-$k$ tetrahedron}.

Analogously to Lemma 4, we can imagine stacking tetrahedrons on four-dimensional space; arranging level-$k$ tetrahedron of factorial basis elements successively below each other, starting from $k = 0$, and $k = 1$ beneath that, $k = 2$ beneath that, and so on. Lemma 8 tells us that if we have a homogeneous polynomial $p$ of degree $k$, represented in the factorial basis as

$$p(t, x, y, z) = \sum_{a+b+c+d=k} \lambda_{a,b,c,d} t^a x^b y^c z^d,$$

then for each monomial $t^a x^b y^c z^d$, the Laplacian applied to that monomial sends the coefficient $\lambda_{a,b,c,d}$ in the level-$k$ tetrahedron to the four positions corresponding to the monomials $t^{a-2} x^b y^c z^d$, $t^a x^{b-2} y^c z^d$, $t^a x^b y^{c-2} z^d$, and $t^a x^b y^c z^{d-2}$ in the level-$(k-2)$ tetrahedron 2 levels above it. Similarly, this leads to the conclusion that $p$ is harmonic precisely when in the level-$k$ tetrahedron representation of $p$, the sum of all coefficients in every subtetrahedron of side 2 is zero.

Since every homogeneous harmonic polynomial is a sum of monomials, we can get a tetrahedral representation for every homogeneous harmonic polynomial. For example, if $p(x, y, z, t) = x^2 - 4xy + 2y^2 - 3z^2 + 4yz - 2t^2$, then in terms of factorial monomials,

$$p(t, x, y, z) = 2x^2 - 4xy + 4y^2 - 6z^2 + 4yz - 4t^2,$$

and hence the tetrahedral representation of $p$ would be represented in the following two-dimensional representation of a three-dimensional tetrahedron of coefficients, see Figure (4.9) below.

\subsection*{4.2.1 Complex Factorial Basis}

Introduce four new variables that will play a crucial role in understanding zonal harmonic polynomials in $\mathbb{A}^4$:

$$r = \frac{x + iy}{2}, \quad s = \frac{x - iy}{2} = \overline{r}, \quad u = \frac{z + it}{2}, \quad \text{and} \quad v = \frac{z - it}{2} = \overline{u}.$$  

Since

$$x = r + s, \quad y = -i(r - s), \quad z = u + v, \quad \text{and} \quad t = -i(u - v),$$
it is clear that any homogeneous polynomial in \( x, y, z, \) and \( t \) can be rewritten as a homogeneous polynomial of the same degree in \( r, s, u, \) and \( v; \) and vice versa. That means

\[
\{ r^a s^b u^c v^d : a + b + c + d = k \quad \text{and} \quad a, b, c, d \in \mathbb{Z} \}
\]

forms a basis of \( P_k(\mathbb{A}^4) \), and we will call it the complex factorial basis.

Here is the crucial Lemma that reveals the usefulness of our new basis. The Laplacian applied to any monomial on \( S^3 \) is also simple.

**Lemma 9** For any integer \( a, b, c, d, \)

\[
\Delta r^a s^b u^c v^d = r^{a-1} s^{b-1} u^{c-1} v^{d-1}.
\] (4.16)

**Proof.** Note that

\[
\frac{\partial^2}{\partial x^2} (r^a s^b u^c v^d) = \frac{u^c v^d}{ablcl!} \frac{\partial^2}{\partial x^2} (r^a s^b) = \frac{u^c v^d}{ablcl!} \left( \frac{a(a-1)}{4} r^{a-2} s^b + \frac{ab}{2} r^{a-1} s^{b-1} + \frac{b(b-1)}{4} r^a s^{b-2} \right)
\]

and by the same principle,

\[
\frac{\partial^2}{\partial y^2} (r^a s^b u^c v^d) = \frac{u^c v^d}{ablcl!} \left( -\frac{a(a-1)}{4} r^{a-2} s^b + \frac{ab}{2} r^{a-1} s^{b-1} - \frac{b(b-1)}{4} r^a s^{b-2} \right)
\]
so that

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (r^a s^b u^c v^d) = \frac{abu^c v^d}{a!b!c!d!} r^{a-1} s^{b-1} u^{c-1} v^{d-1} = r^{a-1} s^{b-1} u^{c-1} v^{d-1}.
\]

Similarly, we will have

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2} \right) (r^a s^b u^c v^d) = r^a s^b u^{c-1} v^{d-1},
\]

so that

\[
\Delta (r^a s^b u^c v^d) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2} \right) (r^a s^b u^c v^d)
= r^{a-1} s^{b-1} u^{c-1} v^{d-1} + r^a s^b u^{c-1} v^{d-1},
\]

as desired. ■

Suppose that we have a homogeneous polynomial \( p \in P_k (\mathbb{A}^4) \) and we construct its tetrahedron of coefficients in the complex factorial basis. Now, each coefficient is associated with the vector \((a, b, c, d)\) corresponding to the power of the monomial it is attached to. If we regard the tetrahedron of coefficients as the tetrahedron of the corresponding vectors, then from Lemma 9, \( p \) is harmonic if and only if in the direction of the vector \((1, 1, -1, -1)\) the entries have constant absolute value and are alternating in sign as we move along the direction of the vector. Thus the basis elements of \( H_k (\mathbb{A}^4) \) correspond to one-dimensional arrays along the direction of vector \((1, 1, -1, -1)\) in the level-\( k \) tetrahedron and consist of alternating plus and minus ones.

To see what the harmonic homogeneous polynomials of degree \( k \) look like, we imagine its tetrahedron of vectors where all the coefficients are initiated to be all 1 first. Starting from the \( rsu \) and \( rs v \) face, for each point in those two faces we attach the vector \((1, 1, -1, -1)\) and see whether the endpoint is still contained in the tetrahedron. If so, then the coefficient is changed to be the different sign from the starting point. If the endpoint of the vector is not in the tetrahedron, the sign of the starting point does not change. Eventually all of the endpoints will have to fall off in the \( rwu \) and \( suv \) face. We now look for some examples for several values of \( k \).

**Example 13** In the case \( k = 2 \), the tetrahedron of coefficients in the complex factorial basis will look like Figure (4.10) below.
Figure 4.10: Level 2 tetrahedron in complex factorial basis

In this case, the basis for $H_2(A^4)$ is given by

$$\{r^2, s^2, u^2, v^2, ru, sv, ru, su, rs - uv\}.$$  

Note that $\dim H_2(A^4) = 9 = (2 + 1)^2$.

**Example 14** In the case $k = 3$, the tetrahedron of coefficients in the complex factorial basis will look like Figure (4.11) below.

So the basis for $H_3(A^4)$ is given by

$$\left\{r^3, s^3, u^3, v^3, r^2v, ruv^2, r^2u, ru^2, s^2u, svu, s^2v, su^2, sv^2, r^2s - ruv, rs^2 - suv, rsu - u^2v, rsuv - uv^2\right\}.$$  

Also note that $\dim H_3(A^4) = 16 = (3 + 1)^2$.

**Example 15** Below we give a basis for $H_k(A^4)$ for several values of $k$. For
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Figure 4.11: Level 3 tetrahedron in complex factorial basis

\( k = 4 \), a basis for \( H_4(\mathbb{A}^4) \) is

\[
\left\{ \begin{array}{l}
  r^4, \quad r^3s - r^2uv, \quad r^2s^2 - rsuv + u^2v^2, \quad rs^3 - s^2uv, \quad s^4, \quad r^3u, \quad r^2u^2, \quad ru^3, \quad u^4, \\
  sv^3, \quad s^2v^2, \quad sv^2 - u^3v, \quad r^2su - ru^2v, \quad rs^2u - su^2v, \quad r^3v, \quad r^2v^2, \quad rv^3, \quad v^4, \\
  sv^3, \quad s^2v^2, \quad sv^2 - u^3v, \quad r^2sv - ru^2v, \quad rs^2v - su^2v
  \end{array} \right. 
\]

Based on these two examples, \( \dim H_k(\mathbb{A}^n) \) seems to be a square number given by \((k + 1)^2\), and this is justified from (3.11). To see this geometrically, note that we need to count the number of lines in the direction of vector \((1, 1, -1, -1)\) that are contained inside the level-\(k\) tetrahedron. To count this, we only need to count the number of points in the \(ruv\) and \(swu\) face. The number of such points is

\[
\text{(#of points in the } ruv \text{ face)} + \text{( #of points in the } swu \text{ face)} - \text{( #of points in the } uv \text{ edge)}.
\]

Since on each face we have \(1 + 2 + \cdots + (k + 1) = \frac{1}{2}(k + 1)(k + 2)\) such points...
and on each edge we have \((k + 1)\) points, then the number of lines is
\[
\frac{1}{2}(k + 1)(k + 2) + \frac{1}{2}(k + 1)(k + 2) - (k + 1) = (k + 1)(k + 2) - (k + 1) = (k + 1)^2.
\]
This is well-known, as in fact these harmonics turn out to span the matrix coefficients of the \((k + 1)\) irreducible representations of \(SU(2) \simeq S^3\).

### 4.3 Zonal Harmonics in the Factorial Basis

We want to consider zonal homogeneous polynomials in the factorial basis, where zonal in this case means rotationally invariant. For this purpose it will be convenient to label the entries of level-\(k\) tetrahedron of basis elements. The rows in the vertical direction will be indexed by \(l = 0, 1, 2, \ldots, k\) with \(t^k\) in row 0. In row \(l\), we will have a triangle with \(x^l t^{k-l}\) at the bottom left of this triangle, \(y^l t^{k-l}\) at the bottom right of the triangle, and \(z^l t^{k-l}\) at the top of the triangle. Again, we can index the basis elements in this triangle. The rows will be indexed by \(m = 0, 1, 2, \ldots, l\) with \(z^l t^{k-l}\) in row 0 and \(x^l t^{k-l}, x^{l-1} y^{k-l}, \ldots, y^{l-1} t^{k-l}, y^l t^{k-l}\) in row \(l\). The elements of the \(m\)th row will finally be indexed by \(n = 0, 1, \ldots, m\) with \(x^n y^{m-n} z^{l-m} t^{k-l}\) in the \(n\)th position. That basis element will be assigned to have coordinates \((l, m, n)\). In other words, the element in the \((l, m, n)\) position is precisely \(x^n y^{m-n} z^{l-m} t^{k-l}\), where \(0 \leq n \leq m \leq l \leq k\).

Now those basis elements whose coordinates \((l, m, n)\) are all even integers form a subtetrahedron of spacing 2 which includes the vertical top term \(t^k\), and we will call those basis elements the even subtetrahedron in the level-\(k\) tetrahedron.

**Theorem 19 (Zonal Harmonic Polynomials on \(S^3\) in Factorial Basis)** Up to a scalar, there is a unique zonal harmonic polynomial of degree \(k\), namely

\[
p_k(x) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=0}^{l} \sum_{n=0}^{m} (-1)^l \frac{l!(2l - 2m)! (2m - 2n)! (2n)!}{(2l + 1)! (l - m)! (m - n)! n!} x^{2n} y^{2m-2n} z^{2l-2m} t^{k-2l}.
\]

**Remark 7** Note that the coefficients are independent of \(k\). We can say that the coefficient corresponding to the coordinates \((2l, 2m, 2n)\) of the zonal harmonic polynomial of degree \(k\) is

\[
a_{(2l, 2m, 2n)} = (-1)^l \frac{l!(2l - 2m)! (2m - 2n)! (2n)!}{(2l + 1)! (l - m)! (m - n)! n!}.
\]
Proof of Theorem 19. Since the only rotationally invariant homogeneous polynomials in \(x, y, \text{ and } z\) are \((x^2 + y^2 + z^2)^l\) up to a scalar, then the zonal harmonic polynomial must be of the form

\[
p(x, y, z, t) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \lambda_{2l} (x^2 + y^2 + z^2)^l t^{k-2l}.
\]

Now if we expand the above expression, we will get

\[
\sum_{l \in \mathbb{Z}} \lambda_{2l} \frac{(x^2 + y^2 + z^2)^l}{l!} t^{k-2l} = \sum_{l \in \mathbb{Z}} \sum_{m=0}^{l} \lambda_{2l} \left( \frac{l}{m} \right) \frac{(x^2 + y^2)^m z^{2l-2m}}{l!} t^{k-2l}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{m=0}^{l} \lambda_{2l} \left( \frac{l}{m} \right) \frac{(x^2 + y^2)^m z^{2l-2m}}{l!} t^{k-2l}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{m=0}^{l} \sum_{n=0}^{m} \lambda_{2l} \left( \frac{l}{m} \right) \left( \frac{m}{n} \right) \frac{x^{2n} y^{2m-2n} z^{2l-2m}}{l!} t^{k-2l}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{m=0}^{l} \sum_{n=0}^{m} \lambda_{2l} \frac{l!}{(l-m)!m!(m-n)!n!} \frac{x^{2n} y^{2m-2n} z^{2l-2m}}{p_k l!} t^{k-2l}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{m=0}^{l} \sum_{n=0}^{m} \lambda_{2l} \frac{x^{2n} y^{2m-2n} z^{2l-2m}}{(l-m)!(m-n)!n!} t^{k-2l}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{m=0}^{l} \sum_{n=0}^{m} \lambda_{2l} \frac{(2l - 2m)!(2m - 2n)!(2n)!}{(l-m)!(m-n)!n!} t^{k-2l}
\]

In order for \(p\) to be harmonic, we must have

\[
0 = \lambda_{2l} \frac{(2l - 2m)!(2m - 2n)!(2n)!}{(l-m)!(m-n)!n!} + \lambda_{2(l+1)} \frac{(2l - 2m)!(2(m+1) - 2n)!(2n)!}{(l-m)!(m+1-n)!n!}
\]

\[
+ \lambda_{2(l+1)} \frac{(2l + 1 - 2m)!(2m - 2n)!(2n)!}{(l+1-m)!(m-n)!n!} + \lambda_{2(l+1)} \frac{(2l - 2m)!(2m - 2n)!(2(n+1))!}{(l-m)!(m-n)!(n+1)!}
\]

which simplifies to

\[
\lambda_{2l} = -\frac{1}{2(2l+1)} \lambda_{2(l-1)}, \text{ for all integers } l \geq 1.
\]
4. Theory of Harmonic Polynomials: Algebraic Approach

By normalizing \( \lambda_i \) such that \( \lambda_0 = 1 \), we will get

\[
\lambda_{2l} = (-1)^l \frac{l!}{(2l+1)!},
\]

which will give us the notation of \( p_k(x) \) in Theorem 19.

Here are the first five values of \( p_k \).

\[
\begin{align*}
p_0 &= 1, \\
p_1 &= t, \\
p_2 &= t^2 - \frac{1}{3}x^2 - \frac{1}{3}y^2 - \frac{1}{3}z^2, \\
p_3 &= t^3 - \frac{1}{3}x^2t - \frac{1}{3}y^2t - \frac{1}{3}z^2t, \\
p_4 &= t^4 - \frac{1}{3}x^2t^2 - \frac{1}{3}y^2t^2 - \frac{1}{3}z^2t^2 + \frac{1}{5}x^4 + \frac{1}{5}y^4 + \frac{1}{5}z^4 \\
&\quad + \frac{1}{15}x^2y^2 + \frac{1}{15}x^2z^2 + \frac{1}{15}y^2z^2.
\end{align*}
\]

Note that we can express \( p_k \) in terms of \( t \) alone because \( p_k \) is a function of \( t \) and \( x^2 + y^2 + z^2 = 1 - t^2 \). Take for example, \( p_2 = t^2 - \frac{1}{3}x^2 - \frac{1}{3}y^2 - \frac{1}{3}z^2 \). Note that

\[
\begin{align*}
p_2 &= t^2 - \frac{1}{3}x^2 - \frac{1}{3}y^2 - \frac{1}{3}z^2 \\
&= \frac{1}{2}t^2 - \frac{1}{6}(x^2 + y^2 + z^2) \\
&= \frac{1}{2}t^2 - \frac{1}{6}(1 - t^2) \\
&= \frac{1}{6}(4t^2 - 1).
\end{align*}
\]

Here are the first five values of \( p_k \) in terms of the variable \( t \).

\[
\begin{align*}
p_0 &= 1, \\
p_1 &= t, \\
p_2 &= \frac{1}{6}(4t^2 - 1), \\
p_3 &= \frac{1}{6}(2t^3 - t), \\
p_4 &= \frac{1}{120}(16t^4 - 12t^2 + 1).
\end{align*}
\]

It turns out that the polynomials \( p_k \) have a nice description as the Chebyshev polynomials of the second kind, denoted by \( U_k(t) \), up to a certain scalar. We have the following Theorem.
4. Theory of Harmonic Polynomials: Algebraic Approach

**Theorem 20** Let $U_k(t)$ be the $k$th Chebyshev polynomial of the second kind and $k \geq 0$ is an integer. Then

$$p_k(t) = \frac{1}{(k+1)!} U_k(t);$$

that is, $p_k(t)$ is proportional to the $k$th Chebyshev polynomial of the second kind.

**Proof.** One way to show that the claim is indeed true, we can show that for each $k \geq 0$, $p_k(t)$ takes one of Chebyshev polynomial of the second kind’s direct form. Now, simple computations show that

$$q = (k + 1)! p_k(t)$$

$$= \sum_{l=0}^{\infty} (-1)^l \left( \frac{1}{2l+1} \right) \frac{(k+1)!}{(k-2l)!} \frac{(x^2+y^2+z^2)^l t^{k-2l}}{(2l+1)! (k-2l)! (x^2+y^2+z^2)^l t^{k-2l}}$$

$$= \sum_{l=0}^{\infty} (-1)^l \left( \frac{k+1}{2l+1} \right) \left( 1-t^2 \right)^l t^{k-2l},$$

which is one of the explicit forms of the Chebyshev polynomials of the second kind $U_k(t)$ (see Weisstein [29, p234], Abramowitz [1, Chapter 22], Oldham [23, Chapter 22], and Rivlin [25]).

Now that we have established that the zonal harmonic polynomials on $S^3$ are nothing other than the Chebyshev polynomials of the second kind, we can find the generating function of the zonal harmonic polynomials in $S^3$. Let us consider the generating function

$$\sum_{k=0}^{\infty} p_k(t) w^k = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} U_k(t) w^k. \tag{4.20}$$

**Theorem 21** The generating function of $p_k(t)$ is given by

$$\sum_{k=0}^{\infty} p_k(t) w^k = e^{tw} \text{sinhc} \left( \sqrt{t^2 - 1}w \right),$$

where $|w| \leq 1$. 

Proof. It is known that the exponential generating function of $U_k(t)$ can be given as

$$\sum_{k=0}^{\infty} \frac{U_k(t)}{k!} w^k = e^{tw} \left( \cosh \left( w \sqrt{t^2 - 1} \right) + \frac{t}{\sqrt{t^2 - 1}} \sinh \left( w \sqrt{t^2 - 1} \right) \right), \text{ for } |w| \leq 1 \quad (4.21)$$

(see Rivlin [25]). Integrating both sides of (4.21) gives us

$$\sum_{k=0}^{\infty} \frac{U_k(t)}{(k+1)!} w^{k+1} = \sum_{k=0}^{\infty} \left( \int_0^w \frac{U_k(t)}{k!} q^k dq \right)$$

$$= \int_0^w \left( \sum_{k=0}^{\infty} \frac{U_k(t)}{k!} q^k \right) dq$$

$$= \int_0^w e^{tq} \left( \cosh \left( q \sqrt{t^2 - 1} \right) + \frac{t}{\sqrt{t^2 - 1}} \sinh \left( q \sqrt{t^2 - 1} \right) \right) dq$$

$$= \int_0^w e^{tq} \cosh \left( q \sqrt{t^2 - 1} \right) dq$$

$$+ \frac{t}{\sqrt{t^2 - 1}} \int_0^w e^{tq} \sinh \left( q \sqrt{t^2 - 1} \right) dq.$$

The first integral could be solved by integration by parts, and so the last equation above reduces to

$$\frac{e^{tq}}{\sqrt{t^2 - 1}} \sinh \left( q \sqrt{t^2 - 1} \right) \bigg|_0^w - \frac{t}{\sqrt{t^2 - 1}} \int_0^w e^{tq} \sinh \left( \sqrt{t^2 - 1}q \right) dq$$

$$+ \frac{t}{\sqrt{t^2 - 1}} \int_0^w e^{tq} \sinh \left( \sqrt{t^2 - 1}q \right) dq$$

$$= \frac{e^{tw}}{\sqrt{t^2 - 1}} \sinh \left( \sqrt{t^2 - 1}w \right).$$

Dividing by $w \neq 0$ in both sides gives us

$$\sum_{k=0}^{\infty} p_k(t) w^k = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} U_k(t) w^k$$

$$= \frac{e^{tw}}{\sqrt{t^2 - 1}w} \sinh \left( \sqrt{t^2 - 1}w \right)$$

$$= e^{tw} \sinhc \left( \sqrt{t^2 - 1}w \right), \quad (4.22)$$
where $|w| \leq 1$. ■

Note the more compact description of the generating function formula for $p_k(t)$. We treat the function above simply as a formal power series giving $p_k(t)$.

### 4.4 Zonal Harmonics in the Complex Factorial Basis

Now it is interesting to actually express (4.19) in terms of the complex factorial basis. For example,

$$p_1 = t = -i(u - v)$$

which is proportional to $u - v$. A more complicated attempt would be to substitute the variables $t, x, y,$ and $z$ in $p_2 = \frac{1}{3}u^2 - \frac{1}{5}(x^2 + y^2 + z^2)$ in terms of $r, s, u, v$. Simple substitution shows that

$$p_2 = -\frac{2}{3}u^2 + \frac{2}{3}(uv - rs) - \frac{2}{3}v^2$$

$$= -\frac{4}{3}u^2 + \frac{2}{3}(uv - rs) - \frac{4}{3}v^2$$

which, up to a scalar, is proportional to

$$u^2 - \frac{1}{2}(uv - rs) + v^2.$$  

Another computation shows that $p_3$ is proportional to

$$u^3 - \frac{1}{3}(uv^3 - rsv) + \frac{1}{3}(uv^2 - rsv) - v^3,$$

while $p_4$ is proportional to

$$u^4 - \frac{1}{4}(uv^3 - rsv) + \frac{1}{6}(uv^2 - rsvu) + \frac{1}{2}(uv - rsv) + v^4.$$

If we see the pattern above, we spot the terms in the Leibniz triangle, one variant of the Pascal triangle. The $l$th elements in the $k$th row, $k = 0, 1, 2, 3, \ldots$ and $l = 0, 1, 2, \ldots k$, given by $a_{k,l}$, represent the coefficients $x^iy^{k-l}$ in $(x + y)^k$ which is $\binom{k}{k}$. In the Leibniz triangle, the $l$th element in the $k$th row is given by

$$a_{k,l} = \frac{1}{\binom{k}{l}}.$$
The terms in $p_k$ are exactly the harmonic polynomials that correspond to the lines with the direction vector $(1, 1 - 1, -1)$ that pass through the edge with $u^k$ and $v^k$ as the endpoints. For example, consider the case where $k = 4$. In the level 4 tetrahedron represented in the complex factorial basis, there are only 5 lines out of 25 lines that pass through the edge with endpoints $u^4$ and $v^4$. Those five are the point $u^4$ itself (the line that passes through this one point in the direction of $(1, 1, 1, -1)$ does not pass any other points), the line that passes through $u^3v$ which also passes through $rsu^2$ (so it corresponds to the harmonic polynomial $u^3v - rsu^2$), the line that passes through $u^2v^2$ which also passes through $rsuv$ and $r^2s^2v^4$ (so it corresponds to the harmonic polynomial $u^2v^2 - rsvu + r^2s^2$), the line that passes through $uv^3$ which also passes through $rsu^2$ (so it corresponds to the harmonic polynomial $uv^3 - rsu^2$), and the point $v^4$ itself (the line that passes through this one point in the direction of $(1, 1, -1, -1)$ does not pass any other points). The coefficient of the expression

$$u^{k-l}v^l - rsu^{k-l-1}v^{l-1} + \cdots + (-1)^l r^ls^ju^{k-2l}$$

is exactly $a_{k,l}$.

We have the following Theorem.

**Theorem 22** The zonal harmonic polynomials $p_k$ described in factorial basis is a complex multiple of

$$p_k = \sum_{l=0}^{k} \binom{k}{l} (-1)^l \left( \sum_{m=0}^{l} (-1)^m \frac{r^m s^m u^{k-l-m} v^{l-m}}{k! m! (k-l-m)! (l-m)!} \right), \quad (4.23)$$

the zonal harmonic polynomials in complex factorial basis.

**Proof.** We need to show that the polynomial $p_k$ given above is a complex multiple of the $U_k(t)$, $k$th Chebyshev polynomial of the second kind. First, note that we can expand (4.23) above as

$$p_k = \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^{l+m} \frac{(k-l)!!}{k! m! (k-l-m)! (l-m)!} r^m s^m u^{k-l-m} v^{l-m}$$

$$= \frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^{l+m} \frac{(k-l)!!}{m! (k-l-m)! (l-m)!} (rs)^m u^{k-l-m} v^{l-m} \quad (4.24)$$

Since $t^2 + x^2 + y^2 + z^2 = 1$ on $S^3$, it is equivalent with saying that $rs + uv = \frac{1}{4}$. 


Hence, Equation (4.24) can be rewritten as

\[ p_k = \frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(-1)^{l+m} (k-l)!!}{m!m! (k-l-m)! (l-m)!} \left( \frac{1}{4} - uv \right)^m u^{k-l-m} v^{l-m} \]

\[ = \frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{l} \sum_{n=0}^{m} \frac{(-1)^{l+m} (k-l)!!}{m!m! (k-l-m)! (l-m)!} \left( \frac{1}{4} \right)^n (-uv)^{m-n} u^{k-l-m} v^{l-m} \]

\[ = \frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{l} \sum_{n=0}^{m} \frac{(-1)^{l+n} (k-l)!!}{4^m m! (k-l-m)! (l-m)! (m-n)! n!} \left( \sum_{m=0}^{n} \frac{1}{m! (k-l-m)! (l-m)! (m-n)!} \right). \quad (4.25) \]

If we change the order of summation in (4.25) above, we get that

\[ p_k = \frac{1}{k!} \sum_{l=0}^{k} \sum_{n=0}^{l} \sum_{m=0}^{n} \frac{(-1)^{l+n} (k-l)!!}{4^n n!} u^{k-l-n} v^{l-n} \]

\[ = \frac{1}{k!} \sum_{l=0}^{k} \sum_{n=0}^{l} \frac{(-1)^{l+n} (k-l)!!}{4^n n!} u^{k-l-n} v^{l-n} \]

\[ \times \left( \sum_{m=0}^{n} \frac{1}{m! (k-l-m)! (l-m)! (m-n)!} \right). \quad (4.26) \]

Now, the term in the bracket in Equation (4.26) above is denoted by \( A(n,l) \), then we have that

\[ A(n,l) = \sum_{m=0}^{l-n} \frac{1}{(m+n)! (k-l-m-n)! (l-m-n)! m!} \]

\[ = \frac{1}{n! (k-l-n)! (l-n)!} \sum_{m=0}^{l-n} \frac{n! (k-l-n)! (l-n)!}{(m+n)! (k-l-m-n)! (l-m-n)! m!}. \]

Here, again we follow the algorithm in Petkovšek [24, Chapter 3] to express \( A(n,l) \) in terms of hypergeometric function. We take out the first term in the summation above so that the first term in

\[ \sum_{m=0}^{l-n} \frac{n! (k-l-n)! (l-n)!}{(m+n)! (k-l-m-n)! (l-m-n)! m!} \]

is 1. Let

\[ a_m = \frac{n! (k-l-n)! (l-n)!}{(m+n)! (k-l-m-n)! (l-m-n)! m!}. \]
and note that
\[
\frac{a_{m+1}}{a_m} = \frac{(m+n)! (k-l-m-n)! (l-m-n)! m!}{(m+n+1)! (k-l-m-n-1)! (l-m-n-1)! (m+1)!} = \frac{(k-l-m-n) (l-m-n)}{(m+n+1) (m+1)} = \frac{(m-(k-l-n)) (m-(l-n))}{(m+n+1) (m+1)}.
\]

so
\[
A(n,l) = \frac{1}{n! (k-l-n)! (l-n)!} 2 F_1 \left( -(k-l-n), -(l-n); n+1; 1 \right).
\]

Again, by using Gauss’ identity (see Gradshteyn [16, Chapter 9] or Andrews [4, Chapter 2]) given in (4.11), let
\[
a = -(k-l-n) = n+l-k, \\
b = -(l-n) = n-l, \\
c = n+1.
\]

Since \( c-a-b = k+1-n > 0 \), then
\[
A(n,l) = \frac{1}{n! (k-l-n)! (l-n)!} \frac{\Gamma (k-n+1) \Gamma (n+1)}{\Gamma (k-l+1) \Gamma (l+1)} = \frac{1}{n! (k-l-n)! (l-n)! (k-l)!!} (k-n)! = \frac{(k-n)!}{(k-l-n)! (l-n)! (k-l)!!}.
\]

Substituting Equation (4.27) to (4.26) gives us
\[
p_k = \frac{1}{k!} \sum_{l=0}^{k} \sum_{n=0}^{l} \frac{(-1)^{l+n} (k-l)!!}{4^n n!} u^{k-l-n} v^{l-n} \left( \frac{(k-n)!}{(k-l-n)! (l-n)! (k-l)!!} \right)
= \frac{1}{k!} \sum_{l=0}^{k} \sum_{n=0}^{l} \frac{(-1)^{l+n} (k-n)!}{4^n n! (k-l-n)! (l-n)!} u^{k-l-n} v^{l-n}.
\]

If we change the order of summation in (4.28), we have that
\[
p_k = \frac{1}{k!} \sum_{n=0}^{k} \sum_{l=n}^{k} \frac{(-1)^{l+n} (k-n)!}{4^n n! (k-l-n)! (l-n)!} u^{k-l-n} v^{l-n}.
\]
Now look at our domain. We can think of that as the pair \((\lambda, \mu)\) where \(0 \leq \mu \leq \lambda \leq \kappa\), and all \(\mu, \lambda, \) and \(\kappa\) are integers. Recall that we define \(\frac{1}{k!}\) to be zero if \(k\) is negative. Since \(k - l - n\) can be negative if \(l\) and \(n\) close enough to \(k\) (say, for example, if \(n = l = k\)), we can remove \((k, k)\) from our domain of summation. After removing all such pairs, we can rewrite Equation (4.29) as

\[
p_k = \frac{1}{k!} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=0}^{k-n} \frac{(-1)^{l+n} (k-n)!}{4^n n! (k-l-n)! (l-n)!} u^{k-l-n} v^{l-n}
\]

\[
= \frac{1}{k!} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^n (k-n)!}{4^n n!} \left( \sum_{l=0}^{k-2n} \frac{(-1)^l (k-2n)!}{(k-l-2n)! n!} u^{k-l-2n} v^l \right)
\]

\[
= \frac{1}{k!} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-n)!}{4^n n! (k-2n)!} \left( \sum_{l=0}^{k-2n} (-1)^l (k-2n)! (k-l-2n)! u^{k-l-2n} v^l \right)
\]

\[
= \frac{1}{k!} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{2^{2n}} \binom{k-n}{n} (u-v)^{k-2n}
\].

(4.30)

Since we have that

\[
u = \frac{z + it}{2} \quad \text{and} \quad v = \frac{z - it}{2},
\]

\(u - v = it\). Substituting this value to (4.30), we get

\[
p_k = \frac{1}{2^{2n} k!} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} 2^{k-2n} \binom{k-n}{n} (it)^{k-2n}
\]

\[
= \left( \frac{i}{2} \right)^k \frac{1}{k!} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^n \binom{k-n}{n} (2t)^{k-2n}
\]

\[
= \left( \frac{i}{2} \right)^k \frac{1}{k!} U_k(t).
\]

(4.31)

Here, we use another representation of Chebyshev polynomials of the second
kind $U_k(t)$ as

$$U_k(t) = \left\lfloor \frac{k}{2} \right\rfloor \sum_{n=0}^{\lfloor k/2 \rfloor} (-1)^n \binom{k-n}{n} (2t)^{k-2n}.$$ 

See Weisstein [29, p234] or Abramowitz [1, Chapter 22].

Equation (4.31) shows that $p_k$ is a complex multiple of $U_k(t)$. From Theorem 20, it has been shown that the zonal harmonic polynomials in factorial is a multiple of $U_k(t)$, so our result follows. ■

Note that Theorem 22 gives us another way of viewing the aforementioned Chebyshev polynomials of the second kind.

To close this chapter, we can think of $p_k$ as the character of the Lie group $S^3$. Recall that a character of the group $G$ is a trace of an irreducible representation (see Alperin [2, Chapter 6], James [20]). Hence the characters of $S^3$ are Chebyshev polynomials of the second kind.
Appendix A

Spherical Harmonic Decomposition of $L^2 (S^{n-1})$

In this appendix we are going to show that the set of all square-integrable functions on $S^{n-1}$ has a natural decomposition over the usual field $F = \mathbb{R}$. Define the Hilbert space $L^2 (S^{n-1})$ which consists of all square-integrable functions on $S^{n-1} \subseteq \mathbb{R}^n$ equipped with the inner product

$$\langle f, g \rangle = \int_{S^{n-1}} f \overline{g} \, d\mu,$$

where $\mu$ is the surface area measure on $S^{n-1}$. We want to find a natural orthogonal decomposition of $L^2 (S^{n-1})$.

Now, let $H$ be any complex Hilbert space. Recall from standard Hilbert theory (see Axler [6, Theorem 5.12]) that we can write

$$H = \bigoplus_{k=0}^{\infty} H_k$$

precisely when $H_k$ is a closed subspace of $H$ for every $k$, $H_k$ is orthogonal to $H_m$ for $k \neq m$, and for every $x \in H$ there exist $x_k \in H_k$ such that $x = x_0 + x_1 + \cdots$, with the sum converging in the norm of $H$. When all of these conditions are met, we say that $H$ is the direct sum of the spaces $H_k$. We have the following important theorem about the decomposition of $L^2 (S^{n-1})$.

**Theorem 23 (Decomposition of $L^2 (S^{n-1})$)**

$$L^2 (S^{n-1}) = \bigoplus_{k=0}^{\infty} H_k (S^{n-1}).$$
A. Spherical Harmonic Decomposition of $L^2(S^{n-1})$

**Proof.** The first condition is met because each $H_k(S^{n-1})$ is finite dimensional and hence is a closed subspace of $L^2(S^{n-1})$. The second condition also holds since we have already established that $H_k(S^{n-1})$ and $H_m(S^{n-1})$ are orthogonal as long as $k \neq m$. For the third condition, we are going to show that the linear span of $$\mathcal{H} = \bigcup_{k=0}^{\infty} H_k(S^{n-1})$$ is dense in $L^2(S^{n-1})$. From there it can be concluded that for every $f \in L^2(S^{n-1})$ there exist $f_k \in H_k(S^{n-1})$ such that $$f = \sum_{k=0}^{\infty} f_k.$$ To see this, note that if we have a homogeneous polynomial $p$ of degree $k$ then from (3.7) we have $$p|s = p_k|s + p_{k-2}|s + \cdots + p_{k-2m}|s,$$ so any polynomial $q$, defined on $A^n$, when restricted on $S^{n-1}$, will be just the sum of finitely many elements in $\mathcal{H}$. By Stone-Weierstrass Theorem (see Rudin [26, Theorem 7.33]) , the set of polynomials $p$ defined in $A^n$ restricted to $S^{n-1}$ is dense in $C(S^{n-1})$, the space of all continuous functions in $S^{n-1}$ with respect to the supremum norm. Since we know that $C(S^{n-1})$ is dense in $L^2(S^{n-1})$ and $L^2$-norm is never greater than the supremum norm, we conclude that $\mathcal{H}$ is dense in $L^2(S^{n-1})$, thus implying that the third condition is met. □

**A.1 Functions on $S^1$ and Fourier Theory**

By the decomposition above, we know that every square integrable function $f$ on $L^2(S^1)$ can be written as $f_0 + f_1 + f_2 + \cdots$ where each $f_i \in H_i(S^1)$. We know that $H_0(S^1) = \{1\}$ and $H_k(S^1) = \langle u^k, v^k \rangle = \langle u^k, v^k \rangle$ for $k \geq 1$. By writing $u$ and $v$ in polar coordinates, we have that $$u = \frac{x + iy}{2} = \frac{1}{2} r e^{i\theta} = \frac{1}{2} e^{i\theta}$$ and $$v = \frac{x - iy}{2} = \frac{1}{2} r e^{-i\theta} = \frac{1}{2} e^{-i\theta}.$$
Hence we have $H_k(S^1) = \langle u^k, v^k \rangle = \langle e^{ik\theta}, e^{-ik\theta} \rangle$ so we can write

$$L^2(S^1) = (1) + \bigoplus_{k=1}^{\infty} \langle e^{ik\theta}, e^{-ik\theta} \rangle = \bigoplus_{k=-\infty}^{\infty} \langle e^{ik\theta} \rangle.$$ 

In particular, any $f \in L^2(S^1)$ can be written as

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$

which is nothing other than standard Fourier theory. Due to the decomposition being orthogonal, the coefficients $a_k$ can be computed by

$$a_k = \int_{S^1} f(\theta) e^{-ik\theta} \, d\mu$$

where, as usual, $\mu$ is the (normalized) measure in $S^1$. The formula above can be found in one of Fourier’s most famous books [18] that is the notation that we use nowadays in the standard Fourier theory. See also Stein [27] or Deitmar [12, Chapter 1] for a more detailed explanation of Fourier theory from the classical point of view.
Appendix B

Integrating Polynomial over a Sphere

In this appendix, we would turn our attention to what actually happens when we integrate a polynomial over a sphere. Although this has been studied extensively for decades, see for instance the papers from Baker [7, page 43] and Folland [17], we would like to present a slightly different derivation of a key formula. Both papers by Baker and Folland make use of the gamma function $\Gamma$ to integrate a homogeneous polynomial (of degree $k$) $p(x)$ over the sphere $S^{n-1}$. In this appendix, we would like to redo the integration and describe the result in terms of a double factorial which we will introduce later. Our general result in (B.15) is aligned with the results of Baker and Folland but the result can be computed efficiently as it is numerically faster to process with computers without having to resort to the gamma function $\Gamma$. Also keep in mind that with this tools in hand, we are able to approximate the integral of any square integrable function defined on $S^{n-1}$, by Appendix A.

Since any polynomial is a sum of homogeneous polynomials, we can assume that $p$ is a homogeneous polynomial of degree $m$. We would like to integrate the polynomial $p$ over $S^{n-1}$. To do this, see that we only need to integrate the monomial $x^\alpha$ where $|\alpha| = m$, since any homogeneous polynomial of degree $m$ is a linear combination of finitely many monomials of the form $x^\alpha$. Let us see what happens in the case where $n = 1$ and 2 first.
B.1 Integrating monomials over $S^1$

Suppose we have $p(x, y) = x^{m_1} y^{m_2}$ where $m_1 + m_2 = m$ and $0 \leq m_1, m_2 \leq m$. We are interested in finding

$$\int_{S^1} x^{m_1} y^{m_2} \, d\mu$$

as a function of $m_1$ and $m_2$, so let’s call it $f(m_1, m_2)$. Here $\mu$ is the normalised measure on $S^1$, i.e. measure on $S^1$ such that

$$\int_{S^1} d\mu = 1.$$

To do that, we can parametrize $S^1$ as follows: write $x = \cos \theta$ and $y = \sin \theta$. By choosing

$$d\mu = \frac{1}{2\pi} d\theta,$$

one gets

$$f(m_1, m_2) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{m_1} \theta \sin^{m_2} \theta \, d\theta. \quad (B.1)$$

Now, fix nonnegative integers $m_1$ and $m_2$, and let

$$P(\theta) = \cos^{m_1+1} \theta \sin^{m_2-1} \theta.$$

Note that

$$\frac{dP}{d\theta} = -(m_1 + 1) \cos^{m_1} \theta \sin^{m_2} \theta + (m_2 - 1) \cos^{m_1+2} \theta \sin^{m_2-2} \theta$$

$$= -(m_1 + 1) \cos^{m_1} \theta \sin^{m_2} \theta + (m_2 - 1) \cos^{m_1} \theta \sin^{m_2-2} \theta (1 - \sin^2 \theta)$$

$$= -(m_1 + m_2) \cos^{m_1} \theta \sin^{m_2} \theta + (m_2 - 1) \cos^{m_1} \theta \sin^{m_2-2} \theta.$$

By the above equation and the Fundamental Theorem of Calculus, we get that

$$0 = \frac{P(2\pi) - P(0)}{2\pi} = -(m_1 + m_2) f(m_1, m_2) + (m_2 - 1) f(m_1, m_2 - 2), \quad (B.2)$$

which translates into the recursive relation

$$f(m_1, m_2) = \frac{m_2 - 1}{m_1 + m_2} f(m_1, m_2 - 2). \quad (B.3)$$
The value $f(m_1, m_2)$ will depend on the parity of $m_2$. If $m_2$ is odd, then $f(m_1, m_2) = C f(m_1, 1)$ where $C$ is some constant which depends on $m_1$ and $m_2$. However, since

$$f(m_1, 1) = \frac{1}{2\pi} \int_0^{2\pi} \cos^{m_1} \theta \sin \theta \, d\theta$$

$$= -\frac{1}{2\pi (m_1 + 1)} \cos^{m_1 + 1} \theta \bigg|_0^{2\pi}$$

$$= 0,$$

then $f(m_1, m_2) = 0$ if $m_2$ is odd. However, when $m_2$ is even, we have

$$f(m_1, m_2) = \frac{(m_2 - 1) \times (m_2 - 3) \times \cdots \times 1}{(m_1 + m_2) \times (m_1 + m_2 - 2) \times \cdots \times (m_2 + 2)} f(m_1, 0)$$

$$= \frac{(m_2 - 1) \times (m_2 - 3) \times \cdots \times 1}{(m_1 + m_2) \times (m_1 + m_2 - 2) \times \cdots \times (m_2 + 2)} \left( \frac{1}{2\pi} \int_0^{2\pi} \cos^{m_1} \theta \, d\theta \right).$$

Let

$$I_k = \frac{1}{2\pi} \int_0^{2\pi} \cos^k \theta \, d\theta,$$

and note that by writing $u = \cos^{k-1} \theta$ and $dv = \cos \theta \, d\theta$ and doing integration by parts, we have the recurrence relation

$$kI_k = (k - 1) I_{k-2}, \text{ for } k \geq 2,$$

and $I_0 = 1, I_1 = 0$. Hence, if $k$ is odd, then $I_k = 0$, and if $k$ is even, we then have

$$I_k = \frac{(k - 1) \times (k - 3) \times \cdots \times 1}{k \times (k - 2) \times \cdots \times 2}.$$
It then follows that \( f(m_1, m_2) \) will be nonzero precisely when both \( m_1 \) and \( m_2 \) are even, and the value is given by

\[
f(m_1, m_2) = \frac{(m_2 - 1) \times (m_2 - 3) \times \cdots \times 1}{(m_1 + m_2) \times (m_1 + m_2 - 2) \times \cdots \times (m_2 + 2)} I_{m_1} \]
\[
= \frac{(m_1 + m_2) \times (m_1 + m_2 - 2) \times \cdots \times (m_2 + 2)}{(m_2 - 1) \times (m_2 - 3) \times \cdots \times 1} \times \frac{(m_1 - 1) \times (m_1 - 3) \times \cdots \times 1}{m_1 \times (m_1 - 2) \times \cdots \times 2} \]
\[
= \frac{(m_1 - 1)!!(m_2 - 1)!!}{(m_1 + m_2)!!} \times \frac{(m_1 - 1)!!(m_2 - 1)!!}{m!!} \]
\[
= \frac{(m_1 - 1)!!(m_2 - 1)!!}{m!!}. \tag{B.4}
\]

Here we use the notation of double factorial defined in the beginning of Chapter 4. To sum up, we have the following theorem.

**Theorem 24** Let \( m_1 \) and \( m_2 \) be any non-negative integers such that \( m_1 + m_2 = m \), and \( p(x, y) = x^{m_1} y^{m_2} \) is a monomial defined on \( S^1 \). If \( \mu \) denotes the normalized measure on \( S^1 \) such that

\[
\int_{S^1} d\mu = 1,
\]

then

\[
\int_{S^1} x^{m_1} y^{m_2} d\mu = \frac{(m_1 - 1)!!(m_2 - 1)!!}{m!!}
\]

precisely when both \( m_1 \) and \( m_2 \) are even; and is zero otherwise.

**Remark 8** Note that with this statement, all irrationalities have been removed.

**B.2 Integrating monomials over \( S^2 \)**

In this case, we do the same thing as what we have done previously. Let \( p(x, y, z) = x^{m_1} y^{m_2} z^{m_3} \) be a monomial of degree \( m \); that is, \( m_1 + m_2 + m_3 = m \) and \( 0 \leq m_1, m_2, m_3 \leq m \). We want to establish the following theorem.

**Theorem 25** Let \( m_1, m_2, \) and \( m_3 \) be any non-negative integers such that \( m_1 + m_2 + m_3 = m \), and \( p(x, y, z) = x^{m_1} y^{m_2} z^{m_3} \) is a monomial defined on \( S^2 \). If \( \mu \)
denotes the normalized measure on $S^2$ such that

$$
\int_{S^2} d\mu = 1,
$$

then

$$
\int_{S^2} x^{m_1} y^{m_2} z^{m_3} d\mu = \frac{(m_1 - 1)! (m_2 - 1)! (m_3 - 1)!}{(m + 1)!}
$$

precisely when all $m_1$, $m_2$, and $m_3$ are even; and zero otherwise.

The usual parametrization of $S^2$ is given by

$$
x = \sin \phi \cos \theta, \quad y = \sin \phi \sin \theta, \quad z = \cos \phi,
$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta < 2\pi$. We can choose our measure to be

$$
d\mu = \frac{1}{4\pi} \sin \phi \, d\phi \, d\theta,
$$

and one can get

$$
f(m_1, m_2, m_3) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \cos^{m_3} \phi \sin^{m_1+m_2+1} \phi \cos^{m_1} \theta \sin^{m_2} \theta \, d\phi \, d\theta
$$

$$
= \frac{1}{4\pi} \left( \int_0^\pi \cos^{m_3} \phi \sin^{m_1+m_2+1} \phi \, d\phi \right) \left( \int_0^{2\pi} \cos^{m_1} \theta \sin^{m_2} \theta \, d\theta \right),
$$

which will be nonzero precisely when both $m_1$ and $m_2$ are even. In that case, we have

$$
f(m_1, m_2, m_3) = \frac{1}{2} \frac{(m_1 - 1)! (m_2 - 1)!}{(m_1 + m_2)!} \int_0^\pi \cos^{m_3} \phi \sin^{m_1+m_2+1} \phi \, d\phi.
$$

Now, let us focus on what the integral above evaluates to by considering the integral of the form

$$
I(k_1, k_2) = \int_0^\pi \cos^{k_1} \theta \sin^{k_2} \theta \, d\theta.
$$

Note that the only difference between the above integral and (B.2) is the boundary. However, in both case, if we also let $P(\theta) = \cos^{k_1+1} \theta \sin^{k_2-1} \theta$, we also have
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\[ P(\pi) - P(0) = 0. \] Hence, \( I \) satisfies the same recurrence relation as \( f \). Note that we are interested in finding

\[
\int_0^\pi \cos^m \phi \sin^{m_1+m_2+1} \phi \, d\phi,
\]

and since \( m_1 + m_2 + 1 \) is odd, we must have

\[
\int_0^\pi \cos^m \phi \sin^{m_1+m_2+1} \phi \, d\phi \\
= \frac{(m_1 + m_2) \times (m_1 + m_2 - 2) \times \cdots \times 2}{(m_1 + m_2 + m_3 + 1) \times (m_1 + m_2 + m_3 - 1) \times \cdots \times (m_3 + 3)} \\
\times \int_0^\pi \cos^m \phi \sin \phi \, d\phi \\
= \frac{(m_1 + m_2)!!}{(m_1 + m_2 + m_3 + 1) \times (m_1 + m_2 + m_3 - 1) \times \cdots \times (m_3 + 3) \times (m_3 + 1)} \left(1 - (-1)^{m_3+1}\right)
\]

which is nonzero precisely when \( m_3 \) is even. In that case, the value is given by

\[
\frac{2 (m_1 + m_2)!!}{(m_1 + m_2 + m_3 + 1) \times (m_1 + m_2 + m_3 - 1) \times \cdots \times (m_3 + 3) \times (m_3 + 1)}.
\]

Hence, if all of \( m_1, m_2, m_3 \) are all even, then

\[
f(m_1, m_2, m_3) = \frac{(m_1 - 1)!! (m_2 - 1)!!}{(m_1 + m_2)!!} \\
\times \frac{(m_1 + m_2)!!}{(m_1 + m_2 + m_3 + 1) \times \cdots \times (m_3 + 3) \times (m_3 + 1)} \\
= \frac{(m_1 - 1)!! (m_2 - 1)!! (m_3 - 1)!!}{(m_1 + m_2 + m_3 + 1)!!} \\
= \frac{(m_1 - 1)!! (m_2 - 1)!! (m_3 - 1)!!}{(m + 1)!!},
\]

and is zero otherwise.
B.3 Integrating monomials over a general sphere

Now, we turn our attention to the case where the underlying surface is \( S^{n-1} \subseteq \mathbb{A}^n \). Suppose we have a monomial \( p(x) = p(x_1, x_2, \ldots, x_n) \) given by

\[
p(x) = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}.
\]

We would like to find

\[
\int_{S^{n-1}} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \, d\mu,
\]

where \( \mu \) is a measure defined similarly as before. To do this, we could use the spherical coordinates

\[
\begin{align*}
x_1 &= \cos \phi_1, \\
x_2 &= \sin \phi_1 \cos \phi_2, \\
x_3 &= \sin \phi_1 \sin \phi_2 \cos \phi_3, \\
&\vdots \\
x_{n-1} &= \sin \phi_1 \sin \phi_2 \sin \phi_3 \cdots \sin \phi_{n-2} \cos \phi_{n-1} \\
x_n &= \sin \phi_1 \sin \phi_2 \sin \phi_3 \cdots \sin \phi_{n-2} \sin \phi_{n-1},
\end{align*}
\]

where all \( \phi_i \in [0, \pi] \) for \( i = 1, 2, \ldots, n-2 \) and \( \phi_{n-1} \in [0, 2\pi) \). We then have

\[
d\mu = \frac{1}{\mu(S^{n-1})} |J| \, d\phi_1 d\phi_2 \cdots d\phi_{n-1}
\]

where \( J \) is the Jacobian matrix of the above transformation given in (B.6) and we have used the short hand form \( d\phi \) to denote \( d\phi_1 d\phi_2 \cdots d\phi_{n-1} \). A direct computation reveals that the Jacobian determinant is given by

\[
|J| = \sin^{n-2} \phi_1 \sin^{n-3} \phi_2 \cdots \sin^2 \phi_{n-3} \sin \phi_{n-2} = \prod_{j=1}^{n-2} \sin^{n-j} \phi_j.
\]

Thus, after rewriting the integrand into spherical coordinates, the integral
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becomes

\[
\frac{1}{\mu(S^{n-1})} \int_0^{\pi} \cdots \int_0^{2\pi} \left( \prod_{i=1}^n x_i^{m_i} \right) \left( \prod_{j=1}^{n-2} \sin^{n-j} \phi_j \right) d\phi
\]

\[
= \frac{1}{\mu(S^{n-1})} \int_0^{\pi} \cdots \int_0^{2\pi} \cos^{m_n-1} \phi_{n-1} \sin^{m_n} \phi_{n-1}
\]

\[
\prod_{i=1}^{n-2} (\cos^{m_i} \phi_i \sin^{m_{i+1} \ldots + m_n + (n-1-i)} \phi_i) \ d\phi
\]

\[
= \frac{1}{\mu(S^{n-1})} \left( \prod_{i=1}^{n-2} \int_0^{\pi} \cos^{m_i} \phi_i \sin^{m_{i+1} \ldots + m_n + (n-1-i)} \phi_i d\phi_i \right)
\]

\[
\times \left( \int_0^{2\pi} \cos^{m_n-1} \phi_{n-1} \sin^{m_n} \phi_{n-1} d\phi_{n-1} \right)
\]

We have the following Theorem.

**Theorem 26** For each \( n \), take any \( n \)-tuple of non-negative integers \( (m_1, \ldots, m_n) \) and let \( p(x) = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \) denote the monomial defined in \( \mathbb{A}^n \). Let \( S^{n-1} \) denote the \((n-1)\)-sphere in \( \mathbb{A}^n \) and \( \mu \) the normalised measure defined on \( S^{n-1} \) such that

\[
\int_{S^{n-1}} d\mu = 1,
\]

then

\[
\int_{S^{n-1}} p(x) \ d\mu = \frac{C_{n-1}}{(\deg(p) + n - 2)!!} \prod_{i=1}^n (m_i - 1)!!,
\]

precisely when \( m_i \) is even for all \( i \in \{1, 2, \ldots, n\} \) and is zero otherwise. Here \( C_{n-1} \) is a constant that depends on \( n \).

**Proof.** The proof goes by induction on \( n \). For the base case, consider \( n = 2 \). Previously we have talked about integrating monomials of the form \( p(x) = x_1^{m_1} x_2^{m_2} \) where \( (m_1, m_2) \in \mathbb{Z}^2_{\geq 0} \) defined on \( \mathbb{A}^2 \) over \( S^1 \). We have established that

\[
\int_{S^1} p(x) \ d\mu = \frac{(m_1 - 1)!! (m_2 - 1)!!}{(m_1 + m_2)!!} = \frac{(m_1 - 1)!! (m_2 - 1)!!}{(\deg(p))!!}
\]

for even value of \( m_1 \) and \( m_2 \) and zero otherwise, so this completes our base case. Now assume that the statement is true for some \( n = r \). We are going to consider
the case when \( n = r + 1 \). Based on our assumption, if we have the monomial
\[
p(x) = x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}
\]
for any \( r \)-tuple of nonnegative integers \((m_1, m_2, \ldots, m_r)\)
and we integrate it over \( S^{r-1} \), we would have that

\[
\int_{S^{r-1}} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \, d\mu = \frac{C_{r-1}}{(\deg(p) + r - 2)!!} \prod_{i=1}^{r} (m_i - 1)!!
\]

\[
= \frac{C_{r-1}}{(m_1 + \cdots + m_r + r - 2)!!} \prod_{i=1}^{r} (m_i - 1)!!
\]

precisely when \( m_1, m_2, \ldots, m_r \) are even and

\[
\int_{S^{r-1}} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \, d\mu = 0
\]

if at least one of \( m_i \) is odd. In case of all \( m_i \) are even, if we parametrize \( S^{r-1} \) by
using the (B.6) described above, we have

\[
\frac{1}{\mu(S^{r-1})} \left( \prod_{i=1}^{r-2} \int_{0}^{\pi} \cos^{m_i} \phi_i \sin^{m_i+1+\cdots+m_r+(r-1-i)} \phi_i \, d\phi_i \right)
\]

\[
\times \int_{0}^{2\pi} \cos^{m_{r-1}} \phi_{r-1} \sin^{m_r} \phi_{r-1} \, d\phi_{r-1}
\]

\[
= \frac{C_{r-1}}{(m_1 + \cdots + m_r + r - 2)!!} \prod_{i=1}^{r} (m_i - 1)!!
\]

(B.7)

Now, if \( n = r + 1 \), take any \((r + 1)\)-tuple of non-negative integers \((a_1, a_2, \ldots, a_{r+1})\)
and let \( p(x) = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \) where we have
\[
\int_{S^r} x_1^{a_1} \cdots x_r^{a_r+1} \, d\mu = \frac{1}{\mu(S^r)} \left( \prod_{i=1}^{r-1} \int_0^\pi \cos^{a_i} \phi_i \sin^{a_i+1+\cdots+a_r+1+(r-i)} \phi_i \, d\phi_i \right) \times \left( \int_0^{2\pi} \cos^{a_r+1} \phi_r \sin^{a_r+1} \phi_r \, d\phi_r \right) = \frac{1}{\mu(S^r)} \left( \int_0^\pi \cos^{a_1} \phi_1 \sin^{a_2+\cdots+a_r+1+r-1} \phi_1 \, d\phi_1 \right) \times \left( \prod_{i=2}^{r-1} \int_0^\pi \cos^{a_i} \phi_i \sin^{a_i+1+\cdots+a_r+1+(r-i)} \phi_i \, d\phi_i \right) \times \left( \int_0^{2\pi} \cos^{a_r} \phi_r \sin^{a_r+1} \phi_r \, d\phi_r \right).
\]
(B.9)

Now, note that based on our induction assumption, the expression
\[
\left( \prod_{i=2}^{r-1} \int_0^\pi \cos^{a_i} \phi_i \sin^{a_i+1+\cdots+a_r+1+(r-i)} \phi_i \, d\phi_i \right) \times \int_0^{2\pi} \cos^{a_r} \phi_r \sin^{a_r+1} \phi_r \, d\phi_r
\]
can be thought of as the left hand side of (B.8) related to the polynomial \( q(x) = x_2^{a_2} x_3^{a_3} \cdots x_r^{a_r} \) in \( S^{r-1} \), so the expression above is nothing other than
\[
\frac{C_{r-1}\mu(S^{r-1})}{(a_2 + \cdots + a_r + r - 2)!! \prod_{i=2}^{r+1} (a_i - 1)!!}
\]
if \( a_2, a_3, \ldots, a_{r+1} \) are all even numbers and zero otherwise. If \( a_2, a_3, \ldots, a_{r+1} \) are all even, Equation (B.9) would then become
\[
\int_{S^r} x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r+1} \, d\mu = \frac{1}{\mu(S^r)} \frac{C_{r-1}\mu(S^{r-1})}{(a_2 + \cdots + a_r + r - 2)!! \prod_{i=2}^{r+1} (a_i - 1)!!} \times \left( \int_0^\pi \cos^{a_1} \phi_1 \sin^{a_2+\cdots+a_r+1+r-1} \phi_1 \, d\phi_1 \right).
\]
Now, the integral on the last equation could be simplified based on the parity of
r. If \( r - 1 \) is even, then the last equation reduces to

\[
C_{r-1} \frac{\mu (S^{r-1})}{\mu (S^r)} \left( \prod_{i=2}^{r+1} (a_i - 1)!! \right)
\times \frac{a_2 + \cdots + a_{r+1} + r - 2}{a_1 + \cdots + a_{r+1} + r - 1}
\times \frac{a_2 + \cdots + a_{r+1} + r - 4}{a_1 + \cdots + a_{r+1} + r - 3}
\times \cdots
\times \frac{a_2}{a_1 + 2}
\times \int_0^\pi \cos^{a_1} \phi_1 \, d\phi_1.
\]

The value of the last integral on the expression above is zero if \( a_1 \) is odd, and

\[
\frac{(a_1 - 1) \times (a_1 - 3) \times \cdots \times 1}{a_1 \times (a_1 - 2) \times \cdots \times 2}
\int_0^\pi d\phi_1 = \frac{\pi \times (a_1 - 1)!!}{a_1!!}
\]

if \( a_1 \) is even. Thus, the expression in (B.10) simplifies to

\[
C_{r-1} \frac{\mu (S^{r-1})}{\mu (S^r)} \left( \prod_{i=2}^{r+1} (a_i - 1)!! \right)
\times \frac{\pi \times (a_1 - 1)!!}{(a_1 + \cdots + a_{r+1} + r - 1)!! \left( \prod_{i=1}^{r+1} (a_i - 1)!! \right)}
= \frac{C_r}{(\deg (p) + r - 1)!! \left( \prod_{i=1}^{r+1} (a_i - 1)!! \right)},
\]

proving the induction step for the case that \( r - 1 \) is even. Now, if \( r - 1 \) is odd,
the equation in (B.9) can be simplified into

\[
\frac{C_{r-1\mu}(S^{r-1})}{\mu(S^r)} \frac{(r+1)!!}{(a_2 + \cdots + a_{r+1} + r - 2)!!} \left( \prod_{i=2}^{r+1} (a_i - 1)!! \right)
\times \left( \frac{(a_2 + \cdots + a_{r+1} + r - 2) \times (a_2 + \cdots + a_{r+1} + r - 4) \times \cdots \times 2}{(a_1 + \cdots + a_{r+1} + r - 1) \times (a_1 + \cdots + a_{r+1} + r - 3) \times \cdots \times (a_1 + 3)} \right)
\times \int_0^\pi \cos^{a_1} \phi_1 \sin \phi_1 \, d\phi_1
\]

\[
= \frac{C_{r-1\mu}(S^{r-1})}{\mu(S^r)} \left( \prod_{i=2}^{r+1} (a_i - 1)!! \right)
\times \left( \frac{1}{(a_1 + 1)} \cos^{a_1 + 1} \phi_1 \bigg|_0^\pi \right).
\]

If \(a_1\) is odd, then the above equation will vanish. Otherwise, the above expression
in (B.11) reduces to

\[
\frac{C_{r-1}\mu(S^{r-1})}{\mu(S^r)} \left( \prod_{i=2}^{r+1} (a_i - 1)!! \right)
\]

\[
\times \frac{1}{(a_1 + \cdots + a_{r+1} + r - 1) \times (a_1 + \cdots + a_{r+1} + r - 3) \times \cdots \times (a_1 + 3)} \left( \frac{2}{a_1 + 1} \right)
\]

\[
= \frac{2C_{r-1}\mu(S^{r-1})}{\mu(S^r)} \frac{1}{(\deg(p) + r - 1) \times (\deg(p) + r - 3) \times \cdots \times (a_1 + 1)} \prod_{i=2}^{r+1} (a_i - 1)!!
\]

\[
= \frac{2C_{r-1}\mu(S^{r-1})}{\mu(S^r)} \frac{1}{(\deg(p) + r - 1)!!} \left( \prod_{i=1}^{r+1} (a_i - 1)!! \right)
\]

\[
= \frac{C_r}{(\deg(p) + r - 1)!!} \left( \prod_{i=1}^{r+1} (a_i - 1)!! \right),
\]

also proving the induction step. Hence, the statement is true for the case \( n = r + 1 \), so it is true for all \( n \in \mathbb{N} \).

Note that in the proof previously presented, it implicitly mentions that if \( r - 1 \) is even, then

\[
C_r = \frac{\pi C_{r-1}\mu(S^{r-1})}{\mu(S^r)}
\]

and if \( r - 1 \) is odd then

\[
C_r = \frac{2C_{r-1}\mu(S^{r-1})}{\mu(S^r)}.
\]

Thus we have two recurrence relations that are connected to each other.

Let \( r \) be an odd number and write \( r = 2s + 1 \). If we combine those two identities together, we have that

\[
C_{2s+1} = \frac{\pi C_{2s}\mu(S^{2s})}{\mu(S^{2s+1})} = \frac{\pi \mu(S^{2s})}{\mu(S^{2s+1})} \left( \frac{2C_{2s-1}\mu(S^{2s-1})}{\mu(S^{2s})} \right) = \frac{2\pi \mu(S^{2s-1})}{\mu(S^{2s+1})} C_{2s-1}.
\]

Now, since we know that the surface area of a \((n-1)\)-sphere \( S^{n-1} \) can be explicitly written as

\[
\mu(S^{n-1}) = \frac{2\pi^n}{\Gamma\left(\frac{n}{2}\right)},
\]
where $\Gamma$ denotes the gamma function, then we have

\[
C_{2s+1} = \frac{2\pi}{2\pi^{s+1}} \left( \frac{2\pi^s}{\Gamma(s)} \right) C_{2s-1} \\
= 2\pi \left( \frac{2\pi^s}{\Gamma(s)} \right) \left( \frac{\Gamma(s+1)}{2\pi^{s+1}} \right) C_{2s-1} \\
= \frac{2\Gamma(s+1)}{\Gamma(s)} C_{2s-1} \\
= 2sC_{2s-1},
\]

where in the last equation we have used one of the properties of the gamma function $\Gamma$ that $\Gamma(s+1) = s\Gamma(s)$ for any positive integer $s$. Now, note that $C_1 = 1$ based on the result presented in (B.5). We have the following Lemma.

**Lemma 10** If $r$ is an odd positive integer, then $C_r = (r - 1)!!$.

**Proof.** Write $r = 2s + 1$ for some $s$. The proof is by induction on $s$. The base case when $s = 0$ is trivial since $C_1 = 1 = 0!! = (1 - 1)!!$. Assume that the statement is true for $s = s_0$, that is, $C_{2s_0+1} = (2s_0)!!$. Now, for the case $s = s_0 + 1$, we have

\[
C_{2s_0+3} = C_{2(s_0+1)+1} = 2(s_0 + 1) C_{2(s_0+1)-1} = (2s_0 + 2) C_{2s_0+1} \\
= (2s_0 + 2) (2s_0)!! = (2s_0 + 2)!! = (2(s_0 + 1))!!,
\]

which completes the proof. ■

Based on the above lemma, we can say something as well about the value of $C_r$ when $r$ is even. Let $r$ be even and write $r = 2s$. To see this, note that

\[
C_r = C_{2s} = \frac{2C_{2s-1}\mu(S^{2s-1})}{\mu(S^{2s})} \\
= 2(2s - 2)!! \left( \frac{2\pi^s}{\Gamma(s)} \right) \left( \frac{\Gamma(s + \frac{1}{2})}{2\pi^{s+\frac{1}{2}}} \right) \\
= \frac{2 (r - 2)!! \Gamma(s + \frac{1}{2})}{\Gamma(s)}.
\]

Note another property of gamma function $\Gamma$, called the Duplication Formula (see [4, Theorem 1.5.1]) which states that

\[
\Gamma(s) \Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s} \Gamma\left(\frac{1}{2}\right) \Gamma(2s)
\]
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foa any positive $s > 0$. If we divide both sides by $\Gamma(s)^2$ we get

$$\frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma(s)} = 2^{1-2s}\Gamma\left(\frac{1}{2}\right) \frac{\Gamma(2s)}{\Gamma(s)^2}. \quad (B.14)$$

Substituting (B.14) to (B.12), we get

$$C_r = \frac{2(2s - 2)!!}{\Gamma\left(\frac{1}{2}\right)} \left(2^{1-2s}\Gamma\left(\frac{1}{2}\right) \frac{\Gamma(2s)}{\Gamma(s)}\right)$$

$$= \frac{(2s - 2)!!}{2^{2s-2}} \left(\frac{\Gamma(2s)}{\Gamma(s)^2}\right).$$

Now, since $s$ is an integer, we can simplify the above equation to get

$$C_r = \frac{(2s - 2)!!}{2^{2s-2}} \left(\frac{\Gamma(2s)}{\Gamma(s)^2}\right)$$

$$= \frac{(2s - 2) \times (2s - 4) \times \cdots \times 2}{2^{2s-2}} \left(\frac{(2s - 2)!!}{\Gamma(s)^2}\right)$$

$$= \frac{1}{2^{2s-2}} \frac{(2s - 2) \times (2s - 4) \times \cdots \times 2}{(s - 1) \times (s - 2) \times \cdots \times 1} \left(\frac{(2s - 1)!}{(s - 1)!}\right)$$

$$= \frac{2^{s-1}}{2^{2s-2}} \left(\frac{(2s - 1)!}{(s - 1)!}\right)$$

$$= \frac{(2s - 1)!}{2^{s-1} (s - 1)!}$$

$$= \frac{(r - 1)!}{(r - 2)!!}$$

$$= (r - 1)!!.$$

To sum up on what we just did, we have the following theorem.

**Theorem 27 (Integral of monomials over $S^{n-1}$)** Let $n \geq 2$ be an integer. If $p(x) = x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n}$ be any monomials of degree $m$ in $S^{n-1} \subset \mathbb{R}^n$, and $\mu$ is the normalised measure on $S^{n-1}$ defined in a way such that

$$\int_{S^{n-1}} d\mu = 1,$$

then

$$\int_{S^{n-1}} p(x) \ d\mu = \frac{(n - 2)!!}{(m + n - 2)!!} \prod_{i=1}^{n} (m_i - 1)!! \quad (B.15)$$

precisely when all of the $m_i$ are even integers, and zero otherwise.
Note the simplicity of the result. To compute the integral, one only needs the power of each variables involved in the monomial (hence one can get the degree of the monomial) and the number $n$. All of them are integers so it is pretty easy to compute (B.15) by any computing devices. Even for small value of $m = \deg(p)$ one can find the integral manually. Note also that all irrationalities have been removed.

**Remark 9** In the main theorem of Folland [17, page 447], it is stated that if $p(x) = x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ and $\beta_i = \frac{1}{2}(m_i + 1)$ then our theorem above in (B.15) could also be written as

$$
\int_{S^n} p(x) \, d\mu = \begin{cases} 
\frac{2\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1+\beta_2+\cdots+\beta_n)} & \text{if all of } m_i \text{ are even,} \\
0 & \text{otherwise.}
\end{cases}
$$

It is worth to mention that the definition of $S^{n-1}$ used in [17] is different than what we use, and the measure $\mu$ is not normalized.

The classical integral of a polynomial $p$ over a general sphere is presented in terms of the gamma function $\Gamma$. While our approach is not as sophisticated as Baker’s or Folland’s, our result is a nice reformulation to the classical result, because the expression

$$
\frac{(n-2)!!}{(m+n-2)!!} \prod_{i=1}^{n} (m_i - 1)!!
$$

is a rational number. Moreover, the symmetry in the above result is also preserved.

Now, it is particularly interesting to see whether the formula will still be neat if one changes the monomial basis to the factorial basis. To see this, suppose we are now interested in integrating $q(x) = x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n+1}$ over $S^{n-1}$. One then have

**Theorem 28** If $q(x) = x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ be any monomials of degree $m$ in $S^{n-1} \subset \mathbb{A}^n$, and $\mu$ is the normalised measure on $S^{n-1}$ defined in a way such that

$$
\int_{S^{n-1}} d\mu = 1,
$$

then

$$
\int_{S^{n-1}} q(x) \, d\mu = \frac{(n-2)!!}{(m+n-2)!!} \left( \prod_{i=1}^{n} m_i!! \right)^{-1}
$$
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precisely when all of \( m_i \) are even integers, and zero otherwise.

**Proof.** The proof is straightforward. Note that

\[
\int_{S^{n-1}} q(x) \, d\mu = \frac{1}{m_1!m_2! \cdots m_n!} \int_{S^{n-1}} p(x) \, d\mu
\]

where \( p(x) \) is defined as in previous theorem. Now, if all the \( m_i \) are even, the above equation becomes

\[
\int_{S^{n-1}} q(x) \, d\mu = \frac{(n-2)!!}{(m+n-2)!!} \prod_{i=1}^{n} \left( \frac{(m_i-1)!!}{m_i!} \right)
\]

\[
= \frac{(n-2)!!}{(m+n-2)!!} \prod_{i=1}^{n} \frac{1}{m_i!!}
\]

\[
= \frac{(n-2)!!}{(m+n-2)!!} \left( \prod_{i=1}^{n} m_i!! \right)^{-1},
\]

as desired. ■

In the particular case where \( n = 4 \), by using two previous theorems, if \( p(x) = p(t, x, y, z) = t^a x^b y^c z^d \) is a monomial of degree \( m \); that is, \( 0 \leq a, b, c, d \leq m \) and \( a + b + c + d = m \), then

\[
\int_{S^3} t^a x^b y^c z^d \, d\mu = \frac{2}{(m+2)!!} \left( \frac{1}{a!!b!!c!!d!!} \right)
\]

precisely when \( a, b, c, d \) are all even numbers, zero otherwise. Also, note that if \( a, b, c, d \) are all even,

\[
\int_{S^3} t^a x^b y^c z^d \, d\mu = \frac{2}{(m+2)!!} \left( \frac{1}{a!!b!!c!!d!!} \right).
\]

**Corollary 7** If \( a, b, c, d \) are all even then

\[
\int_{S^3} \Delta t^a x^b y^c z^d \, d\mu = \frac{2}{(m+2)!!} \left( \frac{1}{a!!b!!c!!d!!} \right).
\]
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Proof. The proof is computational. Observe that from 4.15 and by using our previous result, we have

\[
\int_{S^3} \Delta t^a x^b y^c z^d \, d\mu
= \int_{S^3} t^{a-2} x^b y^c z^d + \frac{t^a x^b y^c z^d}{1} + t^a x^b y^c z^{d-2} \, d\mu
= \frac{2}{m!!} \left( \frac{1}{(a-2)!!b!!c!!d!!} + \frac{1}{a!!(b-2)!!c!!d!!} + \frac{1}{a!!b!!(c-2)!!d!!} + \frac{1}{a!!b!!c!!(d-2)!!} \right)
= \frac{2}{m!!} \left( \frac{a}{a!!b!!c!!d!!} + \frac{b}{a!!b!!c!!d!!} + \frac{c}{a!!b!!c!!d!!} + \frac{d}{a!!b!!c!!d!!} \right)
= \frac{2}{m!!} \left( \frac{m}{a!!b!!c!!d!!} \right)
= \frac{2}{(m-2)!!} \left( \frac{1}{a!!b!!c!!d!!} \right),
\]

as desired. ■

In general, if we have a monomial \( p(\mathbf{x}) = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \) of degree \( m \) defined on \( \mathbb{A}^n \), then

\[
\Delta \left( x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \right) = x_1^{m_1-2} x_2^{m_2} \ldots x_n^{m_n} + \ldots + x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n-2},
\]

so we have that

\[
\int_{S^{n-1}} \Delta \left( x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \right) \, d\mu
= \int_{S^{n-1}} x_1^{m_1-2} x_2^{m_2} \ldots x_n^{m_n} + \ldots + x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n-2} \, d\mu
= \int_{S^{n-1}} x_1^{m_1-2} x_2^{m_2} \ldots x_n^{m_n} \, d\mu + \ldots + \int_{S^{n-1}} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n-2} \, d\mu
= \frac{(n-2)!!}{(m-2+n-2)!!} \left( \frac{1}{(m_1-2)!! \ldots m_n!!} + \ldots + \frac{1}{m_1!! \ldots (m_n-2)!!} \right)
= \frac{(n-2)!!}{(m+n-4)!!} \left( \frac{m_1 + m_2 + \ldots + m_n}{m_1!! m_2!! \ldots m_n!!} \right)
= \frac{m (n-2)!!}{(m+n-4)!!} \left( \prod_{i=1}^{n} m_i!! \right)^{-1},
\]

precisely when all of \( m_i \) are even; zero otherwise.

To close this appendix, we have the following Theorem.
Theorem 29 Let \( p(x) = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \) be a monomial of degree \( m \) on \( \mathbb{A}^n \). If \( \mu \) denotes the normalized measure on \( S^{n-1} \) so that
\[
\int_{S^{n-1}} d\mu = 1,
\]
then we have that
\[
\int_{S^{n-1}} \Delta p(x) \ d\mu = m (m + n - 2) \int_{S^{n-1}} p(x) \ d\mu.
\]

Proof. The proof is simple. Assume all of \( m_i \) are zero since if at least one of them is odd, then both sides reduce to zero. Let
\[
A = \int_{S^{n-1}} \Delta p(x) \ d\mu = \frac{m (n-2)!!}{(m+n-4)!!} \left( \prod_{i=1}^{n} m_i!! \right)^{-1}
\]
from our previous observation. Let
\[
B = \int_{S^{n-1}} p(x) \ d\mu = \frac{(n-2)!!}{(m+n-2)!!} \left( \prod_{i=1}^{n} m_i!! \right)^{-1}.
\]
We have that
\[
\frac{A}{B} = \frac{m (m+n-2)!!}{(m+n-4)!!} = m (m + n - 2),
\]
as desired. \( \blacksquare \)
Bibliography


